A Survey of Markov Additive Processes

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1 Definition (seminar talk on 30 May 2003)

Markov additive processes(MAP) are a class of Markov processes which have important applications to queueing models. The state space of an MAP is multidimensional. It is at least two dimensional, so the state space can be splited into two components $E \times F$, one of which, say E, is Markov component and the other, F, is additive component. Corresponding stochastic process can be denoted by (X, Y). Generally speaking, an MAP (X, Y) is a Markov process whose transition probability measure is translation invariant in the additive component Y. We give the definition in the following.

1.1 Definition 1(Cinlar,1972)

Definition: We split the definition into several steps so that we can understand the definition clearly.

- $T = [0, \infty)$ or $T = \{0, 1, 2, \cdots\}$ — Time parameter space;
- $(\Omega, \mathcal{M}, \mathcal{M}_t, P)$ — Probability space;
- $(E, \mathcal{E})$ — a measurable space

  $\{X_t, t \in T\} : \Omega \mapsto E$ — a stochastic process

  $(F, \mathcal{F}) = (\mathbb{R}^m, \mathcal{R}^m)$ — m dimensional Euclidean space

  $\{Y_t, t \in T\} : \Omega \mapsto F$ — a stochastic process

  Let $(X, Y) = \{(X_t, Y_t), t \in T\}$ be a Markov process on $(E \times F, \mathcal{E} \times \mathcal{F})$ with respect to $\{\mathcal{M}_t, t \in T\}$, where transition function is

$$P_{s,t}(x, y; A \times B).$$

  If this function satisfies some certain conditions, then (X, Y) will be called MAP. We introduce these conditions in the following step.

- Let $\{Q_{s,t}, s < t, s, t \in T\}$ be a family of transition probabilities from $(E, \mathcal{E})$ to $(E \times F, \mathcal{E} \times \mathcal{F})$. If

$$Q_{s,t}(x, A \times B) = \int_{E\times F} Q_{s,u}(x, dy \times dz) Q_{u,t}(y, A \times (B - z)).$$

for any $s < u < t, s, u, t \in T$, $x \in E, A \in \mathcal{E}, B \in \mathcal{F}$ and $B + a = \{b + a, b \in B\}$, then $\{Q_{s,t}, s < t, s, t \in T\}$ is called a semi-Markov transition function on $(E, \mathcal{E}, \mathcal{F})$.

- If

$$P_{s,t}(x, y; A \times B) = Q_{s,t}(x, A \times (B - y)).$$

then (X, Y) is a MAP with respect to $\{\mathcal{M}_t, t \in T\}$ and with semi-Markov transition function $Q_{s,t}$.

The above condition means that:

$$P_{s,t}(x, y; A \times B) = P_{s,t}(x, 0; A \times (B - y)).$$

Understanding of the Definition:

- $X(t)$ is a Markov process / Markov Chain in the state space E;

- In any time interval $(s, t]$, $X(\cdot)$ runs freely according its own rule. Under the conditions that the information of $X(\cdot)$ in $(s, t]$ is known, the increment of $Y(\cdot)$ in this interval is independent from itself before.

1.2 Definition 2(Pacheco António & Probhu N.U.)

In many cases, the state space of the Markov component is discrete. For this special case, we give the definition of MAP as the following.

Definition:

The stochastic process $(X, Y) = \{(X(t), Y(t)), t \in T\}$ on $E \times F$ is a MAP if

(i) (X, Y) is a Markov process;

(ii) for $s, t \in T$, the conditional distribution of $X(s+t)$, $Y(s+t) - Y(s)$, given $X(s), Y(s)$, dependents only on $X(s)$.

Understanding:

- The transition probability should satisfy that:

$$\begin{aligned}
&P(X(s+t) = k, Y(s+t) \in A | X(s) = j, Y(s) = y) \\
&= P(X(s+t) = k, Y(s+t) - Y(s) \in A - y | X(s) = j) \\
&= P(X(s+t) = k | X(s) = j) \cdot P(Y(s+t) - Y(s) \in A - y | X(s) = j, X(s+t) = k)
\end{aligned}$$

  In some sense, the middle line gives the joint distribution of the Markov component and Additive component while the last line gives the conditional distribution of the additive component.

- Because (X, Y) is Markov, it follows easily from the above equation that X is Markov and that Y has conditionally independent increments. Since, in general, Y is non-Markovian, this is the reason why we call X the Markov component and Y the additive component of the MAP (X, Y).

- We should give a remark that a marginal process of a multidimensional Markov process need not to be Markovian. The simplest example is in the queueing system, after introducing the compensational variable, with the increment of one dimension, the non-Markovian process becomes an Markov one.

1.3 E is a single point

In this case:

$$\begin{aligned}
P(Y(s+t) \in A | Y(s) = y) &= P(X(s+t) = e, Y(s+t) \in A | X(s) = e, Y(s) = y) \\
&= P(X(s+t) = e, Y(s+t) - Y(s) \in A - y | X(s) = e) \\
&= P(Y(s+t) - Y(s) \in A - y)
\end{aligned}$$

That means: $Y(\cdot)$ is an independent increment process. In some sense, **MAP can be regarded as the extension of the independent increment process**. In general, the additive component of the MAP does not have independent increment.

1.4 $T = \mathbb{N}$

In this case, MAP is Markov Random Walk(MRW).

An stochastic process $\{(X_n, Y_n), n \in N\}$ on $E \times R^m$ is called MRW if its transition probability measure has the property:

$$\begin{aligned}
&P(X_{m+r} = k, Y_{m+r} \in A | X_m = j, Y_m = y)\\
&= P(X_{m+r} = k, Y_{m+r} - Y_m \in A - y | X_m = j)\\
&= P(Y_{m+r} - Y_m \in A - y | X_m = j, X_{m+r} = k) \cdot P(X_{m+r} = k | X_m = j)
\end{aligned}$$

Noting that the additive component can be written as $Y_n = \sum_{l=1}^{n}(Y_l - Y_{l-1})$, which is like a random walk in the usual sense.

1.5 E is finite, $T = \mathbb{N}$

The state space of MAP is $E \times \mathbb{R}^m$ $(m \geq 1)$;

In this case, MAP is specified by the measure-valued matrix(kernel): $F(dx) = (F_{ij}(dx))$;

$Z_n = Y_n - Y_{n-1}$ .... incretement of additive component

$F_{ij}(dx) = P_{i,o}(X_1 = j, Z_1 \in dx)$

$p_{ij} = F_{ij}(\mathbb{R}^m)$......................transition probability of Markov component

$H_{ij}(dx) = P(Z_1 \in dx | X_0 = i, X_1 = j) = \frac{F_{ij}(dx)}{p_{ij}}$

This provides the convenient of simulating a MAP. Since we can simulate the Markov chain first, and then $Z_1, Z_2, \cdots$ by generating Z_n according to H_{ij} when $X_{n-1} = i$, $X_n = j$.

In a particular case, if $m = 1$ and F_{ij} are concentrated on $(0, \infty)$. The MAP is a Markov Renewal Process(MRP).

(X_n, Y_n) is a MRP, in which $Z_n = Y_n - Y_{n-1}$ can be interpreted as interarrival times.

More generally, when the additive component of an MAP takes values in $\mathbb{R}_+^m$, then the MAP is said to be *MRP*. MRPs are thus discrete versions of MAPs with the additive component taking values in $\mathbb{R}_+^m (m \geq 1)$, and they are called *Markov subordinators*.

1.6 E is finite, T is continuous

In this case(see [1]), $X(t)$ is specified by its intensity matrix $\Lambda = (\lambda_{ij})_{i,j \in E}$,

(1) On an interval $[s, s+t)$ where $X_s \equiv i$, Y_s evolves like a Lv́ey process (stationary independent increment) with characteristic triplet $(\mu_i, \sigma_i^2, \nu_i(dx))$ depending on i, i.i. Lévy exponent

$$\kappa^{(i)}(\alpha) = \alpha\mu_i + \alpha^2\sigma_i^2/2 + \int_{-\infty}^{\infty} (e^{\alpha x} - 1 - \alpha x I_{(|x| \leq 1)})\nu_i(dx).$$

(2) A jump of $\{X_t,\}$ from i to $j \neq i$ has probability q_{ij} of giving rise to a jump of $\{Y_t\}$ at the same time, the distribution of which has some distribution B_{ij}.

1.7 E is infinite

A MAP may be much more complicated. As an example, let X_t be standard Brownian motion on the line. Then a Markov additive process can be defined by letting

$$Y_t = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon t} \int_0^t I(|X_s| \leq \varepsilon) \mathrm{d}s.$$

be the local time at 0 up to time t.

1.8 An Example of Markov Chain with Transition Matrix of Repeating Block Structure

There is a Markov chain whose transition matrix is that

$$P = \begin{pmatrix} B_0 & B_1 & B_2 & \cdots \\ A_{-1} & A_0 & A_1 & \cdots \\ 0 & A_{-1} & A_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where A_i is a $S \times S$ matrix, $i \geq -1$. The above transition matrix often appeared as a repeating structure transition probability matrix in applied probability, especially in queueing system. For example, for that of $M/G/1$ type queue system. In order to understand the meaning clearly, suppose that S is finite.

We can grouped the state space into level and in each level, the state is called phase. See the following picture.

Since $\sum_{k=-1}^{\infty} A_k = \overrightarrow{\mathbf{1}}$ which is exactly the transition probability of the Markov component with the state space $E = \{1, 2, \cdots, s\}$. The state space of the additive space is just the collection of "Level".

Since the Markov chain can jump to the one lower level, only after extend the space of the level to the negative axis, we could regard this Markov chain as a MAP.

1.9 Some Remarks

The basic references to MAPs are still Cinlar[5, 6] to which the most important reference is [3]. For recent work on MAPs up to 1991, see [11].

A survey of MRWs and some new results up to 1991 have been given in [14].

The single major reference to MRPs is [4]. A review of the literature on MRPs around 1991, see [11]. Markov subordinators are reviewed in [13], where applications to queueing systems are considered.

2 Transformation invariant properties of MAPs (seminar talk on 11 Sep 2003)

In this section, we give some transformation invariant properties of MAPs. We assume that, if (X, Y) is an MAP, then $Y(0) = 0$ and the MAP is time-homogeneous. Noting that the time parameter is continuous and the state space of the Markov component is discrete. The transition probability measure is that:

$$P_{jk}(A,t) = P(Y(t) \in A, X(t) = k | X(0) = j).$$

In the following, let $\alpha, \beta \in \mathbb{R}$, $A \in \mathcal{R}^m$ and $B \in \mathcal{R}^n$.

- (Linear transformations of MAPs) Suppose that (X,Y) is an MAP on $E \times \mathbb{R}^m$ with transition probability measure P^Y, and $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation. If $Z = T(Y)$, then (X,Z) is an MAP on $E \times \mathbb{R}^n$ with transition probability measure P^Z such that
$$P^Z_{jk}(B;t) = P^Y_{jk}(T^{-1}(B);t).$$

- (Patching together independent MAPs) Suppose that (X_1,Y) and (X_2,Z) are MAPs on $E_1 \times \mathbb{R}^m$ and $E_2 \times \mathbb{R}^n$ with transition probability measures P^Y and P^Z, respectively. If (X_1,Y) and (X_2,Z) are independent, then $((X_1,X_2),(Y,Z))$ is an MAP on $E_1 \times E_2 \times \mathbb{R}^{m+n}$ with transition probability measure $P^{(Y,Z)}$ such that
$$P^{(Y,Z)}_{(j_1,j_2)(k_1,k_2)}(A \times B;t) = P^Y_{j_1,k_1}(A;t)P^Z_{j_2,k_2}(B;t).$$

- (Marginals of MAPs) If (X,Y) is an MAP on $E \times \mathbb{R}^m$ with transition probability measure P^Y and $Z = (Y_{i_1}, Y_{i_2}, \cdots, Y_{i_n})$ with $1 \le i_1 < \cdots < i_n \le m$, then (X,Z) is an MAP on $E \times \mathbb{R}^m$. If $Z = (Y_1, Y_2, \cdots, Y_m)$, then (X,Z) has transition probability measure
$$P^Z_{jk}(B;t) = P^Y_{jk}(B \times \mathbb{R}^{m-n};t).$$

- (Linear combinations of dependent MAPs) If $Y = (Y_1, Y_2, \cdots, Y_m)$, $Z = (Z_1, Z_2, \cdots, Z_m)$, and $(X,(Y,Z))$ is an MAP on $E \times \mathbb{R}^{2m}$ with transition probability measure $P^{(Y,Z)}$, then the process $(X, \alpha Y + \beta Z)$ is an MAP on $E \times \mathbb{R}^m$ with transition probability measure
$$P_{jk}(A;t) = P^{(Y,Z)}_{jk}(\{(x,y) : x,y \in \mathbb{R}^m, \alpha x + \beta y \in A\};t).$$

- (Linear combinations of independent MAPs) If (X_1,Y) and (X_2,Z) are independent MAPs on $E_1 \times \mathbb{R}^m$ and $E_2 \times \mathbb{R}^n$ with transition probability measure P^Y and P^Z, respectively, then $((X_1,X_2), \alpha Y + \beta Z)$ is an MAP on $E_1 \times E_2 \times \mathbb{R}^m$ with transition probability measure
$$P_{(j_1,j_2)(k_1,k_2)}(A;t) = \int_{\{(x,y):x,y\in\mathbb{R}^m, \alpha x+\beta y\in A\}} P^Y_{j_1k_1}(dy;t)P^Z_{j_2k_2}(dz;t)$$

- We say that (X,Y) is a strong MAP if for any stopping time T we have
$$P\{(Y(T+t) - Y(T) \in A, X(T+t) = k | \mathcal{F}^{(X,Y)}_T\} = P_{X(T)k}(A;t).$$

 Suppose (X,Y) is an MAP on $E \times \mathbb{R}^m$, and $T^*_n (n \ge 0)$ are stopping times such that $0 = T^*_0 \le T^*_1 \le \cdots < \infty$ a.s. Define $X^*_n = X(T^*_n)$, and $Y^*_n = Y(T^*_n)$, for

$n \in \mathbb{N}$. If (X, Y) is a strong MAP, then $(T^*, X^*, Y^*) = \{(T_n^*, X_n^*, Y_n^*), n \in \mathbb{N}\}$ is an MRW on $\mathbb{R}_+ \times E \times \mathbb{R}^m$. Moreover, in case the conditional distribution of $(T_{n+1}^* - T_n^*, X_{n+1}^*, Y_{n+1}^* - Y_n^*)$ given X_n^* does not depend on n, then (T^*, X^*, Y^*) is a homogeneous MRW with one step transition probability measure

$$V_{jk}(A \times B) = P\{T_1^* \in A; X(T_1^*) = k, Y(T_1^*) \in B | X(0) = j\},$$

for $j, k \in E$, $A \in \mathcal{R}_+$ and $B \in \mathcal{R}^m$.

3 Matrix Moment Generating Function

We often suppose that the state space of the Markov component is finite, otherwise the situation is very complicated. For a MAP $\{(J_t, S_t), t \in T\}$, let's consider the matrix form of the generating function, i.e. the matrix $\hat{F}_t[\alpha]$ with ijth element $E_i[e^{\alpha S_t}; J_t = j]$.

Theorem 1 *(p311 in [1]) For a MAP (J_t, S_t)*

(1) If time is disctere, then $\hat{F}_n[\alpha] = \hat{F}[\alpha]^n$ where

$$\begin{aligned} \hat{F}[\alpha] &= \hat{F}_1[\alpha] = (E_i[e^{\alpha S_1}; J_1 = j])_{i,j \in E} = (\hat{F}_{ij}[\alpha])_{i,j \in E} \\ &= (p_{ij} \hat{H}_{ij}[\alpha])_{i,j \in E} \end{aligned}$$

(2)If time is continuous, then the matrix $\hat{F}_t[\alpha]$ is given by $e^{tK[\alpha]}$, where

$$K[\alpha] = \Lambda + (\kappa^{(i)}(\alpha))_{diag} + (\lambda_{ij} q_{ij} (\hat{B}_{ij}[\alpha] - 1)).$$

where $\Lambda, \mu_i, \sigma_i^2, \nu(dx), q_{ij}, B_{ij}$ are parameters of the MAP.

4 The Matrix Paradigms GI/M/1 and M/G/1 (p316 in[1])

Assume now that the additive component S_t of the discrete-time MAP $\{(J_n, S_n)\}$ is lattice, i.e. the MAP has state space $E \times Z$. If we define

$$f_{ij}(k) = F_{ij}(\{k\}) = P_{i,0}(J_1 = j, S_1 - S_0 = k),$$

then $F(k) = (f_{ij}(k))$ can be written as the following $(E \times Z) \times (E \times Z)$ matrix (transition matrix for the MAP)

$$\begin{pmatrix} \ddots & & & \vdots & & & \\ & F(0) & F(1) & F(2) & F(3) & F(4) & \\ & F(-1) & F(0) & F(1) & F(2) & F(3) & \\ \cdots & F(-2) & F(-1) & F(0) & F(1) & F(2) & \cdots \\ & F(-3) & F(-2) & F(-1) & F(0) & F(1) & \\ & F(-4) & F(-3) & F(-2) & F(-1) & F(0) & \\ & & & \vdots & & & \ddots \end{pmatrix}$$

by partitioning into $E \times E$ blocks corresponding to levels.

We shall consider some modification $\{(I_n, L_n)\}$ of the MAP $\{(J_n, S_n)\}$ at level 0, let the state space of the Markov component at level 0 be E_0 which maybe different from that of E. Away from level 0,$\{(I_n, L_n)\}$ moves as $\{(J_n, S_n)\}$. However, when the MAP at level $l > 0$ attempts to go to a level < 0, $\{(I_n, L_n)\}$ is reset to level 0 and a phase in E_0 and a phase in E_0 with probabilities depending on l; the jump out of level 0 may have any distribution. The transition matrix can be written in the block-partitioned form as

$$P' = \begin{pmatrix} C & A(1) & A(2) & A(3) & \cdots \\ B(1) & F(0) & F(1) & F(2) & \cdots \\ B(2) & F(-1) & F(0) & F(1) & \cdots \\ B(3) & F(-2) & F(-1) & F(0) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad (1)$$

Where the dimensions are C: $E_0 \times E_0$, $A(k)$: $E_0 \times E$, $B(l)$: $E \times E_0$, $F(l) : E \times E$. Particular important forms are obtained by letting the MAP be right-or left-continuous (skip-free) for levels, i.e. of one of the forms

$$P' = \begin{pmatrix} C & F(1) & 0 & 0 & \cdots \\ B(1) & F(0) & F(1) & 0 & \cdots \\ B(2) & F(-1) & F(0) & F(1) & \cdots \\ B(3) & F(-2) & F(-1) & F(0) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad (2)$$

$$P' = \begin{pmatrix} C & A(1) & A(2) & A(3) & \cdots \\ F(-1) & F(0) & F(1) & F(2) & \cdots \\ 0 & F(-1) & F(0) & F(1) & \cdots \\ 0 & 0 & F(-1) & F(0) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad (3)$$

We say that matrices of the form (2) are of the GI/M/1 type and those of the form (3) of the M/G/1 type. the obvious motivation being the imbedded Markov chains in the queues GI/M/1 and M/G/1.

Theorem 2 *(Proposition 3.1 in p318 [1]) Assume in ([?]) that both P' and $P = \sum_{-\infty}^{\infty} F(k)$ are irreducible and stochastic, and let ν be the stationary distribution of P and $\mu = \sum_{-\infty}^{\infty} kF(k)1$. Then $\{(I_n, L_n)\}$ is recurrent if and only if $\nu\mu \leq 0$, and positive recurrent if and only if (a) $\sum_{-\infty}^{\infty} kA(k)1 < \infty$ and (b) $\nu\mu < 0$.*

For the queueing model of GI/M/1 type, we have

Theorem 3 *(Corollary 3.3 in p319 [1]) Suppose that both P' in (2) and $P = \sum_{-\infty}^{1}$ are irreducible and stochastic, and let ν be the stationary distribution of P and $\mu = \sum_{-\infty}^{1} kF(k)1$. Then recurrence of P' is equivalent to $\nu\mu \leq 0$ and positive recurrent to $\nu\mu < 0$.*

If we define $R(k)$ as the matrix with elements

$$r_{ij} = E_{il} \sum_{n=0}^{C-1} I(I_n = j, L_n = l + k)$$

where $C = \inf\{n \geq 1 : L_n = l\}$. We write $R = R(1) = (r_{ij})$. Let $\pi = (\pi_l)_{l \in N}$, where $\pi_l = (\pi_{il})_{i \in E}$, then

Theorem 4 *(Lemma 3.4 in p320 [1]) (i) The matrices $R(k)$ do not depend on the choice of $l = 0, 1, 2, \cdots$; (ii) $R(k) = R^k$; (iii) in the recurrent case, π is a stationary measure fro $\{(I_n, L_n)\}$ if and only if π_0 is stationary for $\{I_n^{(l)}\}$ and $\pi_k = \pi_0 R^k, k = 1, 2, \cdots$, where $\{I_n^{(l)}\}$ is the phase at the kth return of $\{L_n\}$ to level l.*

Theorem 5 *(Theorem 3.7 in p321 [1]) R is solution to $R = \psi(R)$ where $\psi(R) = F(1) + RF(0) + R^2F(-1) + \cdots$ and the minimal nonnegative solution to this equation. Furthermore, R can be evaluated by successive iterations, say as limit of the nondecreasing sequence $R^{(n)}$ given by $R^{(0)} = 0$, $R^{(n+1)} = \psi(R^{(n)})$.*

5 Wiener-Hopf Factorization (seminar talk on 16 Sep 2003)

([10])In queueing theory Wiener-Hopf techniques were first used in a non-probabilistic context by Smith [15] in 1953, and later in a probabilistic context by Spitzer [16, 17] around 60's. The specific problem investigated by these authors was the solution of Lindley's [8] integral equation (1952).

5.1 Lindley Processes / equation

(p92 in [1]) Consider the GI/G/1 queue and let T_n, U_n, W_n be the interarrival time, service time and waiting time of customer n respectively, $n = 0, 1, \cdots$. What is the relation between W_n and W_{n+1}.

Say that customer n arrives at time t and customer $n+1$ at $t + T_{n+1}$. The residual work in the system is W_n just before t, $W_n + U_n$ just after t. Therefore we have that

$$W_{n+1} = (W_n + U_n - T_{n+1})^+$$

where $x^+ = \max(x, 0)$. Of course, $X_n := U_n - T_{n+1}$ are i.i.d. which means that $\{W_n\}$ is a Lindley process.

Definition 6 *(p92 in [1]) A discrete time process of the form*

$$W_0 = w, \qquad W_{n+1} = (W_n + X_n)^+, \qquad n = 0, 1, \cdots \tag{4}$$

is called a Lindley process, where $X_0, X_1, \cdots$ are i.i.d. say with common distribution K.

Under some conditions (Corollary 6.5 in [1]), let $n \to \infty$ in (4) we get that $W_n \xrightarrow{\mathcal{D}} M$ and $M \stackrel{\mathcal{D}}{=} (M + X)^+$, where X is independent of M with distribution K. Furthermore $H(x) = P(M \le x)$ is the unique distribution function on $[0, \infty)$ which solves Lindley's integral equation

$$H(x) = \int_{0-}^{\infty} H(dy)K(x - y), \qquad (x \ge 0) \tag{5}$$

5.2 Random Walk

5.2.1 Definition

Definition 7 *(p220 in [1]) Let $X_0, X_1, \cdots$ are i.i.d. with common distribution K, then*

$$S_0 = 0, \qquad S_n = X_1 + \cdots + X_n, \qquad n \ge 1$$

is called a random walk.

For above random walk, we define the following parameters (p221 in [1]) :

τ_+ $= \tau_+^s = \inf\{n \ge 1 : S_n > 0\}$, the first (strict) ascending ladder epoch. The distribution of τ_+ may be defective, i.e. $P(\tau_+ = \infty) > 0$.

S_{τ_+} the first (strict) ascending ladder height (defined on $\{\tau_+ < \infty\}$ only).

G_+ the (strict) ascending ladder height distribution $G_+(x) = P(S_{\tau_+} \le x)$. Here G_+ is concentrated on $(0, \infty)$ and may be defective, i.e. $\|G_+\| = P(\tau_+ < \infty) < 1$.

τ_- $= \tau_-^w = \inf\{n \ge 1 : S_n \le 0\}$ the first (weak) descending ladder epoch.

S_{τ_-} the (weak) descending ladder height distribution $G_-(x) = P(S_{\tau_-} \le x)$. Here G_- is concentrated on $(-\infty, 0]$ and may be defective, i.e. $\|G_-\| = P(\tau_- < \infty) < 1$.

Weak ascending and strict descending ladder epochs can also be defined the obvious way by

$$\tau_+^w = \inf\{n \ge 1 : S_n \ge 0\}, \qquad \tau_-^s = \inf\{n \ge 1 : S_n < 0\}.$$

Results on ladder processes and classification can be found in p223-227 of [1].

5.2.2 Wiener-Hopf Factorization of Random Walk

Theorem 8 *(Theorem 3.1 and Corollary 3.2 in p228 of[1])*

(a) $K = G_+ + G_- - G_+ * G_-$*, or in terms of characteristic function:*

$$1 - \hat{K} = (1 - \hat{G}_+)(1 - \hat{G}_-). \tag{6}$$

(b) $K = G_+^w + G_-^s - G_+^w * G_-^s$.

5.2.3 Solving the Lindley's equation

(p82 in [10]) In order to solve the (5), introducing an auxiliary function $\tilde{H}(x)$ such that

$$\tilde{H}(x) = \begin{cases} \int_{0-}^{\infty} H(dy)K(x-y), & x \le 0 \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Taking fourier transforms of (5) and (7) we obtain

$$\hat{H}(w)(1 - \hat{K}(w)) = H(0) - \hat{\tilde{H}}(w). \tag{8}$$

where

$$\hat{H}(w) = \int_{0-}^{\infty} e^{iwx} H(dx), \qquad \hat{\tilde{H}}(w) = \int_{-\infty}^{0+} e^{iwx} \tilde{H}(dx)$$

$$\hat{K}(w) = \int_{-\infty}^{\infty} e^{iwx} K(dx),$$

Using the uniqueness of (6) and (8) we get that

$$\begin{aligned} \hat{H}(w) &= H(0)(1 - \hat{G}_+(w))^{-1} \\ \hat{\tilde{H}}(w) &= H(0)\hat{G}_-(w) \end{aligned}$$

5.3 Symmetric Wiener-Hopf factorization of Probability Measure/Random Walk

The Wiener-Hopf factorization in (6) is an asymmetric one. From now now, let's consider symmetric Wiener-Hopf factorization. This is also based on the random walk.

Let F be a distribution of a probability measure, whose characteristic function is

$$\phi(\zeta) = \int_{-\infty}^{\infty} e^{i\zeta x} F(dx).$$

Let $X_1, X_2, \cdots$ be mutually independent random variables with the common distribution F and let

$$S_0 = 0, \qquad S_n = X_1 + X_2 + \cdots + X_n, \qquad n \ge 1.$$

Obviously, $\{S_n\}$ is a random walk.

On one hand, define

$$N^+ = \inf\{n \ge 1 : S_n > 0\}$$

then the joint distribution is given by

$$H_n^+(A) = P(N = n, S_N \in A) = P(S_1 \leq 0, \cdots, S_{n-1} \leq 0, S_n \in A), \qquad A \subset (0, \infty)$$

Instead of the bivariate characteristic function we introduce the combination of generating and characteristic function

$$\chi^+(s, \zeta) = E(s^N e^{i\zeta S_N}) = \sum_{n=1}^{\infty} s^n \int_0^{\infty} e^{i\zeta x} H_n^+(dx).$$

On the other hand, define

$$N^- = \inf\{n \geq 1 : S_n < 0\}$$

then we also have S_{N^-}, H_n^-, χ^-, respectively.

Besides the definitions above, we also consider the event of a return to the origin through negative values. Its probability distribution $\{F_n\}$ is given by

$$f_n = P(S_1 < 0, \cdots, S_{n-1} < 0, S_n = 0), \qquad n \geq 1.$$

Apparently, we also have

$$f_n = P(S_1 > 0, \cdots, S_{n-1} > 0, S_n = 0), \qquad n \geq 1.$$

We put $f(s) = \sum_{n=1}^{\infty} f_n s^n$, clearly, $\sum f_n \leq P(X_1 < 0) < 1$.

Theorem 9 *(p605 in [7]) For $|s| \leq 1$, one has the identity*

$$1 - s\phi(\zeta) = (1 - \chi^+(s, \zeta))(1 - f(s))(1 - \chi^-(s, \zeta)). \tag{9}$$

Remark 10 *Factorization (6) is a special case of (9). In fact, let $s = 1$ in (9), then we have*

$$1 - \hat{G}_-(t) = (1 - f(1))(1 - \chi^-(1, t)),$$

which means that

$$\hat{G}_-(t) = f(1) + (1 - f(1))\chi^-(1, t),$$

which is exact the relationship of characteristic functions between the weak and strict descending heights of random walk, which is given by Proposition 1.1 in p222 of [1].

5.4 One Example

Factorization (9) can be regarded as a special case of winer-hopf factorization of MAP with the state space of the Markov component is just single point.

Suppose the time index of a MAP (J_n, S_n) is discrete, the state space of the Markove component is a single point, say, $\{e\}$ and that of the additive component is lattice, say, Z. If we define

$$f_{ee}(k) = F_{ee}(\{k\}) = P_{e,0}(J_1 = e, S_1 - S_0 = k) = P(X_1 = k) = \begin{cases} \lambda & \text{if } k = 1, \\ \mu & \text{if } k = -1, \\ 0 & \text{otherwise.} \end{cases}$$

and $\lambda + \mu = 1$, then $\{F(k)\} = \{(f_{ee}(k))\}$ can be written as the following $Z \times Z$ matrix (transition matrix for the MAP).

$$\begin{pmatrix} \ddots & & & \vdots & & & \\ & 0 & \lambda & & & & \\ & \mu & 0 & \lambda & & & \\ \cdots & & \mu & 0 & \lambda & & \cdots \\ & & & \mu & 0 & \lambda & \\ & & & & \mu & 0 & \\ & & & \vdots & & & \ddots \end{pmatrix}$$

This is exactly the MAP extension of the the imbedded Markov chain in the queues of M/M/1.

5.4.1 From the theoretical point of view

Regarding $\{S_n\}$ as a random walk, from (9) with $s = 1$, we have that

$$1 - \phi(t) = (1 - \chi^+(1,t))(1 - f(1))(1 - \chi^-(1,t)). \tag{10}$$

where

$$\phi(t) = \sum e^{itx} K(X_1 = x) = \lambda e^{it} + \mu e^{-it}.$$

Then (10) is that

$$1 - (\lambda e^{it} + \mu e^{-it}) = (1 - \chi^+(1,t))(1 - f(1))(1 - \chi^-(1,t)). \tag{11}$$

Noting that for the discrete case, $S_{N^+} \equiv 1$ and $S_{N^-} \equiv -1$, we have that

$$\chi^+(1,t) = \|G_+\| e^{it}, \qquad \chi^-(1,t) = \|G_-\| e^{-it}.$$

Factorization (11) becomes that

$$1 - (\lambda e^{it} + \mu e^{-it}) = (1 - \|G_+\| e^{it})(1 - f(1))(1 - \|G_-\| e^{-it}). \tag{12}$$

5.4.2 From the application point of view

Regarding $\{J_n, S_n\}$ as a special MAP, then matrix $A_l = f_{ee}(l)$ is scalar, $l \in \mathbf{Z}$. Define ([9])

$$\vartheta_l^- = \inf\{n \geq 1, S_n \leq l\}, \qquad l \in \mathbf{Z}. \tag{13}$$

where ϑ_l^- is the hitting time of S_n at the set $(-\infty, l]$. Define scalar R_l^+, G_l^- and H_0^- by

$$\begin{aligned} R_l^+ &= E(\sum_{n=1}^{\infty} 1(S_n = l, n < \vartheta_{l-1}^- < \infty)|S_0 = 0), \qquad l \geq 1, \\ G_l^- &= P(S_{\vartheta_{-1}^-} = l|S_0 = 0), \qquad l \leq -1, \\ H_0^- &= P(S_{\vartheta_0^-} = 0|S_0 = 0), \end{aligned} \tag{14}$$

R_l^+ is the expected number of visiting the state l before the additive component of the MAP entering $(-\infty, l-1]$, given that it starts from 0 at time 0.

G_l^- is the probability distribution at l at the time that the additive component first entering $(-\infty, -1]$, given that it starts from 0 at time 0 .

H_0^- is the probability of the additive component is 0 at the time when it first entering $(-\infty, 0]$, given that it starts from 0 at time 0.

For the sequence of $\{A_l, l \in \mathbf{Z}\}$, $\{R_l^+, l \geq 1\}$, $\{G_l^-, l \leq -1\}$, we define

$$A_*(z) = \sum_{l=-\infty}^{\infty} A_l z^l, \quad R_*^+(z) = \sum_{l=1}^{\infty} R_l^+ z^l, \quad G_*^-(z) + \sum_{l=-\infty}^{-1} G_l^- z^l. \tag{15}$$

Then we have

Theorem 11 *(Lemma 3.1 in [9])*

$$1 - A_*(z) = (1 - R_*^+(z))(1 - H_0^-)(1 - G_*^-(z)). \tag{16}$$

In our special case, we have that

$$A_l = \begin{cases} \lambda & \text{if } l = 1, \\ \mu & \text{if } l = -1, \\ 0 & \text{otherwise.} \end{cases} \quad R_l^+ = \begin{cases} R_1^+ & \text{if } l = 1, \\ 0 & \text{if } l \geq 2. \end{cases} \quad G_l^- = \begin{cases} G_l^- & \text{if } l = -1, \\ 0 & \text{if } l \leq -2. \end{cases}$$

Factorization (16) becomes that

$$1 - (\lambda z + \frac{\mu}{z}) = (1 - R_1^+ z)(1 - H_0^-)(1 - \frac{G_{-1}^-}{z}). \tag{17}$$

¿From the definition, we know that the descending ladder height distribution G_- and G^- in (14) are the same, and the $f(1)$ and H_0^- in (14) are the same. If regarding (12) as a expression involving the characteristic function and that (17) as a expression involving the generating function, and let $z = e^{it}$, then factorization (12) and that (17) are the same, therefore, we have that

$$R_1^+ = \|G_+\|, \qquad G_{-1}^- = \|G_-\|.$$

In fact, for this special example, the factorization of(17) is that

$$1 - (\lambda z + \frac{\mu}{z}) = \begin{cases} (1 - \frac{\lambda}{\mu} z)(1-\lambda)(1-\frac{1}{z}) & \text{if } \lambda < \mu, \\ (1-z)(1-\mu)(1-\frac{\mu}{\lambda}\frac{1}{z}) & \text{if } \lambda > \mu. \end{cases}$$

Therefore, we have that

$$R_1^+ = \|G_+\| = \begin{cases} \frac{\lambda}{\mu} & \text{if } \lambda < \mu, \\ 1 & \text{if } \lambda > \mu. \end{cases} \qquad G_{-1}^- = \|G_-\| = \begin{cases} 1 & \text{if } \lambda < \mu \\ \frac{\mu}{\lambda} & \text{if } \lambda > \mu. \end{cases}$$

and that

$$f(1) = H_0^- = \begin{cases} \lambda & \text{if } \lambda < \mu \\ \mu & \text{if } \lambda > \mu. \end{cases}$$

5.5 Symmetric Wiener-Hopf factorization in MAP([2])

Definition 12 *Let π be a σ-finite measure on $(E, \mathcal{E})$, we shall say that the MAP's*

$$(J,S) = (J_n, S_n, P^x) \quad \textit{and} \quad (\hat{J}, \hat{S}) = (\hat{J}_n, \hat{S}_n, \hat{P}^x)$$

with semi-Markov transition functions $Q, \hat{Q}$ respectively, are in duality relative to π if

(a) for every $x \in E$, $P(x, \cdot) = Q(x, \cdot \times F) \ll \pi$, $\hat{P}(x,\cdot) = \hat{Q}(x, \cdot \times F) \ll \pi$;

(b) for every $B \in \mathcal{B}(R^m)$, $f, g \in \mathcal{B}(E)_+$,

$$< f, Q(B)g >=< f\hat{Q}(-B), g >$$

where, for $f_1, g_1 \in \mathcal{B}(E)_+$, we have $< f_1, g_1 >= \int f_1(x)g_1(x)\pi(dx)$. In this case we say also that Q and $\hat{Q}$ are in duality relative to π.

Definition 13 *(1) Define the Fourier transform of Q by*

$$(Q(\theta)f)(x) = \int\int Q(x, dx' \times dy) f(x') e^{i(\theta, y)}, \qquad \theta \in F = R^m,$$

where $(\cdot,\cdot)$ denotes the usual inner product in $F = R^m$.

(2) Operators T and T^* is said to be adjoint if $< f, T(B)g >=< fT(-B)^*, g >$ for every $B \in \mathcal{B}(R^m)$, $f, g \in \mathcal{B}(E)_+$.

(3) For any stopping time, define also that

$$(Hf)(x) = (H_N(\tau,\theta))f(x) = E^x(\tau^N e^{i(\theta, S_N)} f(J_N); N < \infty).$$

Then we have

Theorem 14 (Wiener-Hopf factorization) *(p110 in [2]) Let (J, S) and $(\hat{J}, \hat{S})$ be in duality relative to π, $N^+, N^{\cdot+}, N_+$ are stopping times, $N_+ = N^{\cdot+} < N^+$ if $N^{\cdot+} < \infty$ and $N_+ = N^+$ if $N^{\cdot+} = \infty$, under some further conditions, then for $0 \leq \tau < 1$, $\theta \in R^m$:*

$$I - \tau Q(\theta) = (I - \hat{H}^*_{\hat{N}^+}(\tau, \theta))(I - H_{N^{\cdot+}}(\tau, \theta))(I - H_{N^+}(\tau, \theta)) \tag{18}$$

where the middle term is interchangeable with $I - \hat{H}^*_{\hat{N}^{\cdot+}}(\tau, \theta)$, I is the identity operator.

This factorization has a measure form, which is

Theorem 15 (Wiener-Hopf factorization, measure form) *(p111 in [2]) For sutable operator valued measure $H_n^+, H_n^{\cdot+}, \hat{H}_n^+$, $n \geq 1$, we have*

$$(I - \tau Q)(B) = \left(I - \sum_{n=1}^{\infty} \tau^n (\hat{H}_n^+)^*\right) \circ \left(I - \sum_{n=1}^{\infty} \tau^n H_n^{\cdot+}\right) \circ \left(I - \sum_{n-1}^{\infty} \tau^n H_n^+\right)(B). \tag{19}$$

where

(1) "$\circ$" denote the convolution product and "*" the adjoint;

and for $x \in E$, $B \in \mathcal{B}(R^m)$, $f \in L^p$ and $n \geq 1$,

(2) $(H_n^+(B)f)(x) = E^x(f(X_n); N^+ = n, S_n \in B)$;

(3) $(H_n^{\cdot+}(B)f)(x) = E^x(f(X_n); N^{\cdot+} = n, S_n \in B)$;

(4) $(\hat{H}_n^+(B)f)(x) = \hat{E}^x(f(\hat{X}_n); \hat{N}^+ = n, \hat{S}_n \in B)$.

5.5.1 Factorization associated with the Markov component of a MAP

In this subsection, we let the Markov component is a Markov chain and suppress the additive component (or let the state space of the additive component is just a single point). Define the stopping time as

$$\begin{aligned} N^+ &= \inf\{n > 0; J_n > J_0\}; \\ N_+ &= \inf\{n > 0; J_n \geq J_0\}; \\ N^{\cdot+} &= N_+, \text{if } N_+ < N^+ \text{ and } N^{\cdot+} = \infty \text{ otherwise }. \end{aligned} \tag{20}$$

If we write $H_{N^+}(\tau) = H_{N^+}(\tau, 0)$ etc. then we have

Theorem 16 *Let P and $\hat{P}$ be in duality relative to π, and consider the stopping times (20). Then for $0 \leq \tau < 1$ we have*

$$I - \tau P = (I - \hat{H}_{\hat{N}^+}(\tau))(I - H_{N^{\cdot+}}(\tau))(I - H_{N^+}(\tau)). \tag{21}$$

where the middle term is interchangeable with $I - \hat{H}^*_{\hat{N}^{\cdot+}}$.

Example 17 (One-dimensional random walk)

Suppose that $J_n = \sum_{k=1}^{n} Z_k$ *where the* $\{Z_k\}$ *are i.i.d random variables on* R^m *with law* μ.

(1) $\pi(dx) = \lambda(dx)$ *(Lebesgue measure) is* P*-excessive;*

(2) $\hat{P}(x, A) = \hat{\mu}(A - x) := \mu(x - A)$*;*

(3) $(Pf)(x) = E^x(f(J_1)) = \int f(x + x')\mu(dx')$*;*

(4) Let $e(x) = e^{i\theta x}$ *for* $\theta \in R^1$*, then* $(Pe)(x) = \phi(\theta)e(x)$ *with* $\phi(\theta)$ *is the characteristic function of* μ.

(5) $(H_{N^+}e)(0) = \chi^+(\tau, \theta)$, $(H_{N^{\cdot+}}e)(0) = f(\tau)$, $(\hat{H}^*_{\hat{N}^+}e)(0) = \chi^-(\tau, \theta)$.

(6) Let (21) act on $e(x)$ *and evaluate at* $x = 0$ *we get (9) immediately.*

5.5.2 Factorization associated with the Additive component of a MAP

Let $F = R^1$ and define the stoppings time as

$$\begin{aligned} N^+ &= \inf\{n > 0; S_n > 0\}; \\ N_+ &= \inf\{n > 0; S_n \geq 0\}; \\ N^{\cdot+} &= N_+, \text{ if } N_+ < N^+ \text{ and } N^{\cdot+} = \infty \text{ otherwise }. \end{aligned} \tag{22}$$

Theorem 18 *Let* Q *and* $\hat{Q}$ *be in duality relative to* π *and consider the stopping times as in (22), then for* $0 \leq \tau < 1$*, we have*

$$I - \tau Q(\theta) = (I - \hat{H}^*_{\hat{N}^+}(\tau, \theta))(I - H_{N^{\cdot+}}(\tau, 0))(I - H_{N^+}(\tau, \theta)) \tag{23}$$

where the middle term is interchangeable with $I - \hat{H}^*_{\hat{N}^{\cdot+}}(\tau, 0)$.

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