Example 1 *Let $\hat{A}(x)$ be the EGF for the number of permutations with even cycles only. Find a simple expression for $\hat{A}(x)$. Also find a simple expression for the number a_{2n} of permutations of $2n$ elements containing even cycles only.*

Solution *Let $\hat{B}(x)$ be the EGF for the number of permutations consisting of a single even cycle. We note that there are $(2n-1)!$ even cycles of length $2n$, and hence*

$$\hat{B}(x) = \sum_{n\geq 1} \frac{(2n-1)!}{(2n)!} x^{2n} = \frac{1}{2} \sum_{n\geq 1} \frac{1}{n} x^{2n} = \frac{1}{2} \log \frac{1}{1-x^2}.$$

Hence we have

$$\hat{A}(x) = \sum_{k\geq 0} \left(\hat{B}(x)\right)^k / k! = \exp(\hat{B}(x)) = (1-x^2)^{-1/2}.$$

Using

$$(1-4x)^{-1/2} = \sum_{n\geq 0} \binom{2n}{n} x^n,$$

we obtain

$$(1-x^2)^{-1/2} = \sum_{n\geq 0} \binom{2n}{n} (x^2/4)^n.$$

Hence

$$a_{2n}/(2n)! = [x^{2n}](1-x^2)^{-1/2} = \binom{2n}{n} 4^{-n}, \quad a_{2n} = (2n)! \binom{2n}{n} 4^{-n}.$$

1 The Exponential Formula for the Set Construction of Labelled Objects

Many combinatorial objects can be decomposed into basic "components" (or "factors"). For example: a permutation is decomposed into cycles, a graph is decomposed into connected components, a 2-regular graph is decomposed into cycles, a rooted tree is decomposed into subtrees, a polynomial is decomposed into irreducible polynomials, and so on. We will express the generating function of all the objects in terms of the generating function of components.

We first deal with labelled objects constructed from a set of components (the components are not ordered). For example, the cycles of a permutation, classes of a set partitions, and connected components of a graph. Let $\hat{C}(x)$ be the EGF for the components, and $\hat{F}(x)$ be the EGF for all the objects. Then we have

$$\hat{F}(x) = \sum_{n\geq 0} (1/k!) \left(\hat{C}(x)\right)^k = \exp(\hat{C}(x)).$$

We can also introduce the bivariate generating function $F(x,y) = \sum_{n,k} f_{n,k} y^k x^n/n!$, where $f_{n,k}$ is the number of objects of size n with exactly k components. Then we have

$$F(x,y) = \sum_{k\geq 0} y^k (1/k!) \left(\hat{C}(x)\right)^k = \exp(y\hat{C}(x)).$$

Example 2 *Let $f_{n,k}$ be the number of 2-regular graphs with k connected components and n vertices which are labelled by $1, 2, \ldots, n$. Note that the components here are cycles with length at least 3. Since there are $(n-1)!/2$ cycles of length n, the EGF for the components is*

$$\hat{C}(x) = \sum_{n\geq 3} (1/2)(n-1)! x^n/n! = (1/2)\sum_{n\geq 3} x^n/n = (1/2)\left(\ln\frac{1}{1-x} - x - x^2/2\right).$$

Hence the bivariate generating function for labelled 2-regular graphs is

$$\begin{aligned} F(x,y) &= \sum_{n,k} f_{n,k} y^k x^n/n! = \exp(y\hat{C}(x)) \\ &= \exp\left((1/2)y\left(\ln\frac{1}{1-x} - x - x^2/2\right)\right) \\ &= (1-x)^{-y/2} e^{-yx/2 - yx^2/4}. \end{aligned}$$

The total number of labelled 2-regular graphs with n vertices is equal to

$$[x^n]F(x,1) = [x^n](1-x)^{-1/2} e^{-x/2 - x^2/4}.$$

Example 3 *Let $\hat{C}(x)$ be the EGF for labelled connected graphs. Let $f_{n,k}$ be the number of labelled graphs with k connected components and n vertices, and*

$$F(x,y) = \sum_{n,k} f_{n,k} y^k x^n/n!$$

Then we have $F(x,y) = \exp(y\hat{C}(x))$. Setting $x = 1$, we obtain

$$F(x,1) = \exp(\hat{C}(x)).$$

Here $F(x,1)$ is the EGF for all labelled graphs, which is easy to find. Since there are $2^{\binom{n}{2}}$ labelled graphs with n vertices, we have

$$F(x,1) = \sum_{n\geq 0} 2^{\binom{n}{2}} x^n/n!.$$

Hence

$$\hat{C}(x) = \ln\left(\sum_{n\geq 0} 2^{\binom{n}{2}} x^n/n!\right) = x + (1/2)x^2 + (2/3)x^3 + (19/12)x^4 + (91/15)x^5 + \cdots.$$