Research Article

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A class of subsums of Euler’s sum

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Abstract: A class of sums of the type
\[ \sum_{n=1}^{\infty} \frac{1}{n^{2k}} \]
is evaluated, where \( k, m \) and \( r \) are positive integers with \( m \geq 2 \) and \( a_1, \ldots, a_r \) are integers satisfying \( 1 \leq a_1 < a_2 < \cdots < a_r \leq m - 1 \).

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MSC 2010: 11A07, 11A25, 11B25, 11B68, 11F66, 11Y60

1 Introduction

Let \( \mathbb{N} := \{1, 2, 3, \ldots\} \), \( \mathbb{N}_0 := \{0, 1, 2, \ldots\} \) and \( \mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\} \). Let \( \mathbb{Q} \), \( \mathbb{Q} \) and \( \mathbb{R} \) denote the fields of rational numbers, algebraic numbers and real numbers, respectively.

In the eighteenth century Euler proved that
\[ \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1}2^{2k-1}B_{2k}n^{2k}}{(2k)!}, \quad k \in \mathbb{N}, \quad (1.1) \]
where \( B_{\ell} (\ell \in \mathbb{N}_0) \) denotes the \( \ell \)th Bernoulli number. Euler’s formula (1.1) is well known and many proofs of it occur in the literature, see for example [2, 3, 8].

Some subsums of Euler’s sum \( \sum_{n=1}^{\infty} \frac{1}{n^{2k}} \) of the type
\[ \sum_{n=a_1, \ldots, a_r (\text{mod } m)}^{\infty} \frac{1}{n^{2k}}, \quad k, m, r \in \mathbb{N}, \quad m \geq 2, \quad (1.2) \]
where \( a_1, \ldots, a_r \in \mathbb{Z} \) satisfy \( 0 \leq a_1 < a_2 < \cdots < a_r \leq m - 1 \), have been evaluated. One very simple example is
\[ \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1}2^{2k-1}B_{2k}n^{2k}}{m^{2k}(2k)!}, \quad k, m \in \mathbb{N}, \quad (1.3) \]
which follows immediately from (1.1). Another simple example is
\[ \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1}(2^{2k} - 1)B_{2k}n^{2k}}{2(2k)!}, \quad k \in \mathbb{N}, \]
which follows by subtracting (1.3) with \( m = 2 \) from (1.1).

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Recently Navas, Ruiz and Varona [6, p. 34, Proposition 3.5] evaluated the subsum
\[ \sum_{n=1}^{\infty} \frac{1}{n^{2k}}, \quad k, m, s \in \mathbb{N}, \]
for \( m \equiv 1 (\text{mod } 2) \), \( m \geq 3 \) and \( s \in \{1, 2, \ldots, (m-1)/2\} \) in terms of values of trigonometric functions and values of Bernoulli polynomials. They stated that there is a similar evaluation for \( m \equiv 0 (\text{mod } 2) \) but did not give it. We now state their theorem in a form valid for all \( m \in \mathbb{N} \) with \( m \geq 3 \) and all \( s \in \mathbb{Z} \) with \( 2s \not\equiv 0 (\text{mod } m) \), and give a very simple proof of it in Section 2. We recall that the Bernoulli polynomial \( B_n(x) \) \((n \in \mathbb{N}_0, x \in \mathbb{R})\) is defined by
\[ B_n(x) := \sum_{j=0}^{n} \binom{n}{j} B_r x^{n-j}, \]
and we note the properties
\[ B_n(0) = B_n, \quad B_{2k}(x) = B_{2k}(1-x), \quad n, k \in \mathbb{N}_0. \]  \( (1.4) \)

**Theorem 1.1** (Navas, Ruiz and Varona). Let \( k, m \in \mathbb{N} \) with \( m \geq 3 \). Let \( s \in \mathbb{Z} \) be such that \( 2s \not\equiv 0 (\text{mod } m) \). Then
\[ \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1} 2^{k-1} \pi^{2k} m^{k-1}}{m(2k)!} \sum_{j=0}^{m-1} B_{2k}(\mathbb{Q}) \cos(2\pi s j/m). \]
As \( B_{2k}(\mathbb{Q}) \in \mathbb{Q} \) and \( \cos(2\pi s j/m) \in \mathbb{Q} \cap \mathbb{R} \), we see that
\[ \frac{1}{n^{2k}} \sum_{n=1}^{\infty} \frac{1}{n^{2k}} \in \mathbb{Q} \cap \mathbb{R}. \]

Hence a sum of the form (1.2) with
\[ \begin{cases} r \equiv 0 (\text{mod } 2), & 1 \leq a_1 < a_2 < \cdots < a_r \leq m-1, \\ (a_j, m) = 1, & a_{r+1-j} = m - a_j, \quad j = 1, \ldots, r, \end{cases} \]
is a sum of sums of the type given in Theorem 1.1, namely
\[ \sum_{j=1}^{\lfloor m/2 \rfloor} \sum_{n=1}^{\infty} \frac{1}{n^{2k}}, \]
where each \( 2a_j \not\equiv 0 (\text{mod } m) \), and thus is of the form \( a \pi^{2k} \), where \( a \in \mathbb{Q} \cap \mathbb{R} \). We give a class of subsums of this type for which \( a \) can be given explicitly as a rational linear combination of square roots of positive integers, see Theorem 4.7. The idea of such a result is implicit in the work of Shanks and Wrench [7] and our purpose is to make it completely explicit. Two examples are
\[ \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{(8 - 5\sqrt{2} + 4\sqrt{3} - 3\sqrt{6})n^2}{288}, \]
see Corollary 5.10, and
\[ \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{n^2}{49} (3 - \sqrt{7}), \]
see Corollary 5.12.

In Section 2 we prove Theorem 1.1. In Section 3 we define the class of subsums of Euler’s sum that we shall evaluate in Section 4. In Section 4 we make use of Theorem 1.1 to prove our main result (Theorem 4.7). In Section 5 we give some examples illustrating Theorem 4.7.
2 Proof of Theorem 1.1

We make use of the Fourier expansion of the Bernoulli polynomial $B_{2k}(x)$ ($k \in \mathbb{N}$), namely,

$$B_{2k}(x) = \frac{(-1)^{k-1}(2k)!}{2^{2k-1}\pi^{2k}} \sum_{n=1}^{\infty} \frac{\cos 2n\pi x}{n^{2k}}, \quad x \in [0, 1],$$  \hspace{1cm} (2.1)

see for example [1, p. 805]. Appealing to (2.1), we obtain

$$\sum_{j=0}^{m-1} \cos(2\pi j/m) B_{2k}(j/m) = \sum_{j=0}^{m-1} \cos(2\pi j/m) \frac{(-1)^{k-1}(2k)!}{2^{2k-1}\pi^{2k}} \sum_{n=1}^{\infty} \frac{\cos(2\pi j/m)}{n^{2k}}$$

$$= \frac{(-1)^{k-1}(2k)!}{2^{2k-1}\pi^{2k}} \sum_{n=1}^{\infty} \frac{1}{n^{2k}} \sum_{j=0}^{m-1} \cos(2\pi j/m) \cos(2\pi jm/m)$$

$$= \frac{(-1)^{k-1}(2k)!}{2^{2k}\pi^{2k}} \sum_{n=1}^{\infty} \frac{1}{n^{2k}} \sum_{j=0}^{m-1} (\cos(2\pi (n-s)j/m) + \cos(2\pi (n+s)j/m)).$$

Now

$$\sum_{j=0}^{m-1} \cos(2\pi (n \mp s)j/m) = \begin{cases} 
    m & \text{if } n \equiv \pm s \text{ (mod } m), \\
    0 & \text{if } n \not\equiv \pm s \text{ (mod } m),
\end{cases}$$

and $s \not\equiv -s \text{ (mod } m)$ (as $2s \not\equiv 0 \text{ (mod } m)$) so

$$\sum_{j=0}^{m-1} \cos(2\pi jm/m) B_{2k}(j/m) = \frac{(-1)^{k-1}(2k)!m}{2^{2k}\pi^{2k}} \sum_{n=1}^{\infty} \frac{1}{n^{2k}} \sum_{n=1}^{\infty} \frac{1}{n^{2k}}$$

from which the asserted formula follows.

3 A class of subsums of Euler’s sum

We begin with some definitions.

**Definition 3.1.** We call a positive integer $d$ a discriminant if $d$ is not a perfect square and $d \equiv 0 \text{ or } 1 \text{ (mod } 4)$. A discriminant $d$ is called a fundamental discriminant if there is no integer $g > 1$ such that $g^2|d$ and $d/g^2 \equiv 0 \text{ or } 1 \text{ (mod } 4)$. The conductor $f = f(d)$ of a discriminant $d$ is the largest positive integer such that $f^2|d$ and $d/f^2 \equiv 0 \text{ or } 1 \text{ (mod } 4)$. The fundamental discriminant $\Delta = \Delta(d)$ associated with the discriminant $d$ is $\Delta = df^2$, where $f$ is the conductor of $d$.

We emphasize that in this paper we are restricting discriminants to be positive integers. The Kronecker symbol for a discriminant $d$ and a positive integer $n$ is written as $(\frac{d}{n})$. Properties of the Kronecker symbol are given in [4, pp. 304–306]. The Kronecker symbol $(\frac{d}{n})$ is a completely multiplicative function of $n$. Moreover,

$$\left( \frac{d}{n} \right) = \begin{cases} 
    0 & \text{if } (n, d) > 1, \\
    \pm 1 & \text{if } (n, d) = 1.
\end{cases}$$  \hspace{1cm} (3.1)

Also, if $f$ is the conductor of the discriminant $d$ and $\Delta = df^2$ is the fundamental discriminant associated with $d$ then

$$\left( \frac{d}{n} \right) = \begin{cases} 
    0 & \text{if } (n, f) > 1, \\
    \left( \frac{d}{\Delta} \right) & \text{if } (n, f) = 1.
\end{cases}$$  \hspace{1cm} (3.2)

Two further properties of the Kronecker symbol are

$$\left( \frac{d}{n} \right) = \left( \frac{d}{d-n} \right), \quad 1 \leq n \leq d-1,$$  \hspace{1cm} (3.3)
see [4, p. 305, Theorem 3.3] for a proof, and
\[
\sum_{\substack{r=1 \\ (r,d)=1}}^{d-1} \left( \frac{d}{r} \right) = 0. \tag{3.4}
\]

By (3.3) we have
\[
\sum_{\substack{1 \leq r < d/2 \\ (r,d)=1}} \left( \frac{d}{r} \right) = \sum_{\substack{d/2 < r < d/2 \\ (r,d)=1}} \left( \frac{d}{r} \right) = \frac{1}{2} \sum_{\substack{r=1 \\ (r,d)=1}}^{d-1} \left( \frac{d}{r} \right)
\]
as \(d/2\) is not an integer if \(d\) is odd and \((d/2, d) = d/2 > 1\) if \(d\) is even since \(d \geq 8\) in this case. Hence, by (3.4), we deduce
\[
\sum_{\substack{1 \leq r < d/2 \\ (r,d)=1}} \left( \frac{d}{r} \right) = 0. \tag{3.5}
\]

The final property of the Kronecker symbol that we need is the identity
\[
\sum_{\substack{t=1 \\ (t,\Delta)=1}}^{\Delta-1} \left( \frac{\Delta}{t} \right) e^{2\pi nt/\Delta} = \left( \frac{\Delta}{n} \right) \sqrt{\Delta}, \tag{3.6}
\]
which is valid for any positive integer \(n\) and any fundamental discriminant \(\Delta\), see [5, p. 221, Theorem 215]. As \((\frac{\Delta}{\bar{n}})\sqrt{\Delta} \in \mathbb{R}\) we have from (3.6)
\[
\sum_{\substack{t=1 \\ (t,\Delta)=1}}^{\Delta-1} \left( \frac{\Delta}{t} \right) \cos(2\pi nt/\Delta) = \left( \frac{\Delta}{n} \right) \sqrt{\Delta}. \tag{3.7}
\]

Our next three definitions are of quantities that we need in order to be able to state our main result (Theorem 4.7).

**Definition 3.2.** For \(k, m \in \mathbb{N}\) and a discriminant \(d\), we define
\[
P_k(m) := \prod_{p|m} \left( 1 - \frac{1}{p^{2k}} \right)
\]
and
\[
P_k(m, d) := \prod_{p|m} \left( 1 - \left( \frac{d}{p} \right) \frac{1}{p^{2k}} \right) = \prod_{p|m} \left( 1 - \left( \frac{d}{p} \right) \frac{1}{p^{2k}} \right),
\]
where \(p\) runs through the primes satisfying the given conditions.

In particular, we have \(P_k(1) = 1\) and \(P_k(m, d) = 1\) if \(m|d\).

**Definition 3.3.** For \(m \in \mathbb{N}_0\) and a discriminant \(d\), we define
\[
S_m(d) := \sum_{\substack{1 \leq r < d/2 \\ (r,d)=1}} \left( \frac{d}{r} \right)^m.
\]

We note that \(S_0(d) = 0\) by (3.5).

**Definition 3.4.** Let \(k \in \mathbb{N}\) and \(d\) a discriminant. Let \(f\) be the conductor of \(d\) and \(\Delta = d/f^2\) the fundamental discriminant associated with \(d\). We define
\[
H_k(d) := \frac{2}{\Delta^k} P_k(f, \Delta) \sum_{r=0}^{2k-1} \left( \frac{2k}{r} \right) \Delta^r B_r S_{2k-r}(\Delta).
\]

We note that in the sum in \(H_k(d)\) the terms with \(r\) (odd) \(\geq 3\) vanish as \(B_{2n+1} = 0\) for \(n \in \mathbb{N}\).

Our final definition defines the class of congruences in (1.2) that we consider.
Definition 3.5. The set of congruences
\[ n \equiv a_1, \ldots, a_r \pmod{m}, \]
where \( m \) and \( r \) are positive integers with \( m \geq 2 \) and \( a_1, \ldots, a_r \) are integers satisfying \( 1 \leq a_1 < a_2 < \cdots < a_r \leq m - 1 \), is said to be discriminantly determined if there exist \( \epsilon_1 = \pm 1, \ldots, \epsilon_s = \pm 1 \) and discriminants \( d_1, \ldots, d_s \) with no nonempty product equal to a perfect square such that
\[ n \equiv a_1, \ldots, a_r \pmod{m} \quad \text{if and only if} \quad \left( \frac{d_1}{n} \right) = \epsilon_1, \ldots, \left( \frac{d_s}{n} \right) = \epsilon_s. \]
The congruences \( n \equiv 7, 17 \pmod{24} \) are discriminantly determined as
\[ n \equiv 7, 17 \pmod{24} \quad \text{if and only if} \quad \left( \frac{8}{n} \right) = 1, \left( \frac{12}{n} \right) = -1. \]
However the congruence \( n \equiv 1 \pmod{4} \) is not discriminantly determined. Our main result evaluates the sum (1.2) for the class of congruences which are discriminantly determined.

4 Proof of main result

In this section we evaluate some infinite series and then state and prove our main result Theorem 4.7.

Proposition 4.1. Let \( e, k \in \mathbb{N} \). Let \( d \) be a discriminant. Then
\[ \sum_{n=1}^{\infty} \left( \frac{d}{n} \right) \frac{1}{n^{2k}} = \left( \frac{d}{e} \right) \frac{1}{e^{2k}} \sum_{n=1}^{\infty} \left( \frac{d}{n} \right) \frac{1}{n^{2k}}. \]

Proof. We have
\[ \sum_{n=1}^{\infty} \left( \frac{d}{n} \right) \frac{1}{n^{2k}} = \sum_{n=1}^{\infty} \left( \frac{d}{en} \right) \frac{1}{(en)^{2k}} = \sum_{n=1}^{\infty} \left( \frac{d}{e} \right) \frac{1}{n^{2k}} \frac{1}{n^{2k}}, \]
and the asserted result now follows. \( \Box \)

Proposition 4.2. Let \( k, m \in \mathbb{N} \). Let \( d \) be a discriminant. Then
\[ \sum_{e | m} \frac{\mu(e)}{e^{2k}} = P_k(m) \quad \text{and} \quad \sum_{e | m} \mu(e) \left( \frac{d}{e} \right) \frac{1}{e^{2k}} = P_k(m, d), \]
where \( \mu \) denotes the Möbius function.

Proof. We just prove the first formula as the second formula can be proved in a similar manner. As \( \mu(e)/e^{2k} \) is a multiplicative function of the positive integer \( e \), and \( \mu(p) = -1 \) and \( \mu(p^2) = \mu(p^3) = \cdots = 0 \) for any prime \( p \), we have
\[ \sum_{e | m} \frac{\mu(e)}{e^{2k}} = \prod_{p \mid m} \left( 1 + \frac{\mu(p)}{p^{2k}} + \frac{\mu(p^2)}{p^{2k}} + \cdots + \frac{\mu(p^{v_p(m)})}{p^{2k v_p(m)}} \right) = \prod_{p \mid m} \left( 1 - \frac{1}{p^{2k}} \right). \]
The asserted formula now follows by Definition 3.2. \( \Box \)

Proposition 4.3. Let \( k, m \in \mathbb{N} \). Then
\[ \sum_{(n, m) = 1}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1} 2^{2k-1} B_{2k} \pi^{2k}}{(2k)!} P_k(m). \]
Proof. Appealing to (1.3) and Proposition 4.2, we obtain
\[
\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \sum_{e|n} \sum_{m=1}^{\infty} \mu(e) \frac{1}{n^{2k}} = \sum_{e|n} \mu(e) \sum_{m=1}^{\infty} \frac{1}{n^{2k}} = \sum_{e|n} \mu(e) \frac{(-1)^{k-1} 2^{k-1} B_{2k} \pi^{2k}}{e^{2k} (2k)!} = \frac{(-1)^{k-1} 2^{k-1} B_{2k} \pi^{2k}}{(2k)!} \sum_{e|n} \mu(e) e^{2k} = \frac{(-1)^{k-1} 2^{k-1} B_{2k} \pi^{2k}}{(2k)!} P_k(m)
\]
as asserted. 

Proposition 4.4. Let \( k \in \mathbb{N} \). Let \( \Delta \) be a fundamental discriminant. Then
\[
\sum_{n=1}^{\infty} \left( \frac{\Delta}{n} \right) \frac{1}{n^{2k}} = \frac{(-1)^{k-1} 2^{k-1} \pi^{2k}}{(2k)! \Delta^{2k} \sqrt{\Delta}} \sum_{r=0}^{2k-1} \binom{2k}{r} \Delta^r B_r S_{2k-1-r}(\Delta).
\]

Proof. Let \( r_1, \ldots, r_{\phi(\Delta)/2} \) be the integers such that
\[
1 \leq r_1 < \cdots < r_{\phi(\Delta)/2} \leq \Delta - 1, \quad \left( \frac{\Delta}{r_1} \right) = \cdots = \left( \frac{\Delta}{r_{\phi(\Delta)/2}} \right) = 1,
\]
and \( s_1, \ldots, s_{\phi(\Delta)/2} \) the integers such that
\[
1 < s_1 < \cdots < s_{\phi(\Delta)/2} < \Delta - 1, \quad \left( \frac{\Delta}{s_1} \right) = \cdots = \left( \frac{\Delta}{s_{\phi(\Delta)/2}} \right) = -1.
\]
We note that \( \phi(\Delta) \equiv 0 \pmod{4} \) and \( (r_m, \Delta) = (S_m, \Delta) = 1, r_{\phi(\Delta)/2+1-m} = \Delta - r_m, s_{\phi(\Delta)/2+1-m} = \Delta - s_m \) for \( m = 1, 2, \ldots, \phi(\Delta)/2 \). Appealing to (3.1), the theorem of Navas, Ruiz and Varona (Theorem 1.1) and (3.7), we obtain
\[
\sum_{n=1}^{\infty} \left( \frac{\Delta}{n} \right) \frac{1}{n^{2k}} = \sum_{n=1}^{\infty} \left( \frac{\Delta}{n} \right) \frac{1}{n^{2k}} = \sum_{m=1}^{\phi(\Delta)/2} \frac{1}{m^{2k}} - \sum_{m=1}^{\phi(\Delta)/2} \frac{1}{n^{2k}} = \sum_{n=1}^{\phi(\Delta)/4} \frac{1}{n^{2k}} - \sum_{n=1}^{\phi(\Delta)/4} \frac{1}{n^{2k}} = \frac{(-1)^{k-1} 2^{k-1} \pi^{2k} \phi(\Delta)/4}{(2k)! \Delta} \sum_{m=1}^{\Delta-1} B_{2k}(t/\Delta) \cos(2\pi r_m t/\Delta) - \frac{(-1)^{k-1} 2^{k-1} \pi^{2k} \phi(\Delta)/4}{(2k)! \Delta} \sum_{m=1}^{\Delta-1} B_{2k}(t/\Delta) \cos(2\pi s_m t/\Delta) = \frac{(-1)^{k-1} 2^{k-1} \pi^{2k}}{(2k)! \Delta} \sum_{t=0}^{\Delta-1} B_{2k}(t/\Delta) \sum_{m=1}^{\Delta-1} \left( \frac{\Delta}{n} \right) \cos(2\pi u t/\Delta)
\]
By (3.3) and (1.4) we have, for $1 \leq t \leq \Delta - 1$ with $(t, \Delta) = 1$,

$$\left( \frac{\Delta}{t} \right) = \left( \frac{\Delta}{t-\Delta} \right), \quad B_{2k}(t/\Delta) = B_{2k}((\Delta - t)/\Delta).$$

We remark that when $\Delta$ is even we have $(\Delta/2, \Delta) = \Delta/2 \neq 1$ as $\Delta \geq 8$, so $t \neq \Delta/2$. Hence, pairing $t$ and $\Delta - t$, we observe appealing to Definition 3.3

$$\sum_{t=1}^{\Delta-1} \left( \frac{\Delta}{t} \right) B_{2k}(t/\Delta) = 2 \sum_{1 \leq t \leq \Delta/2 \atop (t,\Delta)=1} \left( \frac{\Delta}{t} \right) B_{2k}(t/\Delta)$$

$$= 2 \sum_{1 \leq t \leq \Delta/2 \atop (t,\Delta)=1} \left( \frac{\Delta}{t} \right) \sum_{r=0}^{2k} \binom{2k}{r} B_r \frac{t^{2k-r}}{\Delta^{2k-r}}$$

$$= \frac{2}{\Delta^{2k}} \sum_{r=0}^{2k} \binom{2k}{r} \Delta^r B_r \sum_{1 \leq t \leq \Delta/2 \atop (t,\Delta)=1} \left( \frac{\Delta}{t} \right) t^{2k-r}$$

$$= \frac{2}{\Delta^{2k}} \sum_{r=0}^{2k} \binom{2k}{r} \Delta^r B_r S_{2k-r}(\Delta)$$

$$= \frac{2}{\Delta^{2k}} \sum_{r=0}^{2k-1} \binom{2k-1}{r} \Delta^r B_r S_{2k-r}(\Delta),$$

as $S_0(\Delta) = 0$. The asserted formula now follows. \hfill \qed

**Proposition 4.5.** Let $k \in \mathbb{N}$. Let $d$ be a discriminant. Let $f$ be the conductor of $d$ and $\Delta = df^2$ the fundamental discriminant associated with $d$. Then

$$\sum_{n=1}^{\infty} \left( \frac{d}{n} \right) \frac{1}{n^{2k}} = \frac{(-1)^{k-1} 2^{2k-1} \pi^{2k}}{(2k)! \sqrt{\Delta}} H_k(d).$$

**Proof.** By (3.1) we have

$$\sum_{n=1}^{\infty} \left( \frac{d}{n} \right) \frac{1}{n^{2k}} = \sum_{n=1}^{\infty} \left( \frac{d}{n} \right) \frac{1}{n^{2k}}. \quad (4.1)$$

Replacing $d$ by $\Delta f^2$ in the right-hand sum in (4.1), and noting that $(n, \Delta f^2) = 1$ is equivalent to $(n, \Delta) = (n, f) = 1$, we deduce that

$$\sum_{n=1}^{\infty} \left( \frac{\Delta f^2}{n} \right) \frac{1}{n^{2k}} = \sum_{n=1}^{\infty} \left( \frac{\Delta}{n} \right) \frac{1}{n^{2k}}.$$

By (3.2) we have $(\frac{\Delta f^2}{n}) = (\frac{\Delta}{n})$ for $(n, f) = 1$ so

$$\sum_{n=1}^{\infty} \left( \frac{d}{n} \right) \frac{1}{n^{2k}} = \sum_{n=1}^{\infty} \left( \frac{\Delta}{n} \right) \frac{1}{n^{2k}}.$$

By (3.1) we have $(\frac{\Delta}{n}) = 0$ for $(n, \Delta) > 1$, so

$$\sum_{n=1}^{\infty} \left( \frac{d}{n} \right) \frac{1}{n^{2k}} = \sum_{n=1}^{\infty} \left( \frac{\Delta}{n} \right) \frac{1}{n^{2k}}.$$
Hence
\[
\sum_{n=1}^{\infty} \left( \frac{d}{n} \right) \frac{1}{n^{2k}} = \sum_{n=1}^{\infty} \left( \sum_{e|n} \mu(e) \right) \left( \frac{\Delta}{n} \right) \frac{1}{n^{2k}} = \sum_{e|n} \mu(e) \sum_{n=1}^{\infty} \left( \frac{\Delta}{n} \right) \frac{1}{n^{2k}}.
\]

Appealing to Propositions 4.1, 4.2 and 4.4, as well as Definition 3.4, we obtain
\[
\sum_{n=1}^{\infty} \left( \frac{d}{n} \right) \frac{1}{n^{2k}} = \sum_{e|n} \mu(e) \left( \frac{\Delta}{e} \right) \frac{1}{e^{2k}} \sum_{n=1}^{\infty} \left( \frac{\Delta}{n} \right) \frac{1}{n^{2k}}
\]
\[= P_k(f, \Delta) \sum_{n=1}^{\infty} \left( \frac{\Delta}{n} \right) \frac{1}{n^{2k}}
\]
\[= P_k(f, \Delta) \frac{(-1)^{k-1} 2^{k-1} \pi^{2k}}{(2k)!} \frac{\Delta^{2k} \sqrt{\Delta}}{\Gamma(2k)} \sum_{r=0}^{\infty} (2k-r) \Delta^r B_r S_{2k-r}(\Delta)
\]
\[= \frac{(-1)^{k-1} 2^{k-1} \pi^{2k}}{(2k)!} \frac{\Delta^{2k} \sqrt{\Delta}}{H_k(d),}
\]

which is the asserted result. 

**Proposition 4.6.** Let \( k, m \in \mathbb{N} \). Let \( d \) be a discriminant. Let \( f \) be the conductor of \( d \). Let \( \Delta = df^2 \) be the fundamental discriminant associated with \( d \). Then
\[
\sum_{n=1}^{\infty} \left( \frac{d}{n} \right) \frac{1}{n^{2k}} = \frac{(-1)^{k-1} 2^{k-1} \pi^{2k}}{(2k)!} P_k(m, d) H_k(d).
\]

**Proof.** Appealing to Propositions 4.1, 4.2 and 4.5, we deduce
\[
\sum_{n=1}^{\infty} \left( \frac{d}{n} \right) \frac{1}{n^{2k}} = \sum_{n=1}^{\infty} \left( \sum_{e|n} \mu(e) \right) \left( \frac{d}{n} \right) \frac{1}{n^{2k}}
\]
\[= \sum_{e|n} \mu(e) \sum_{n=1}^{\infty} \left( \frac{d}{n} \right) \frac{1}{n^{2k}}
\]
\[= \sum_{e|n} \mu(e) \left( \frac{d}{e} \right) \frac{1}{e^{2k}} \sum_{n=1}^{\infty} \left( \frac{d}{n} \right) \frac{1}{n^{2k}}
\]
\[= P_k(m, d) \frac{(-1)^{k-1} 2^{k-1} \pi^{2k}}{(2k)!} \frac{\Delta^{2k} \sqrt{\Delta}}{H_k(d),}
\]

which is the asserted result. 

We are now ready to state and prove our main result.

**Theorem 4.7.** Let \( m \) and \( h \) be positive integers with \( m \geq 2 \) and \( a_1, \ldots, a_h \) integers satisfying \( 1 \leq a_1 < a_2 < \cdots < a_h \leq m - 1 \). Suppose that the set of congruences
\[ n \equiv a_1, \ldots, a_h \pmod{m} \]
is discriminantly determined, say by discriminants \( d_1, \ldots, d_r \) (with no nonempty product \( d_{j_1} \cdots d_{j_r} \) (\( 1 \leq j_1 < \cdots < j_r \leq r \)) equal to a perfect square) and \( \epsilon_1 = \pm 1, \ldots, \epsilon_r = \pm 1 \). Then
\[ \sum_{n=a_1,\ldots,a_r \pmod{m}} \frac{1}{n^{2k}} = \frac{(-1)^{k-1} 2^{k-1} \pi^{2k}}{(2k)!} P_k(d_1 \cdots d_r)
\]
\[+ \frac{(-1)^{k-1} 2^{k-1} \pi^{2k}}{(2k)!} \sum_{s=1}^{r} \sum_{1 \leq j_1 < \cdots < j_s \leq r} \epsilon_{j_1} \cdots \epsilon_{j_s} \frac{H_k(d_{j_1} \cdots d_{j_s})}{\Delta(d_{j_1} \cdots d_{j_s})} P_k(d_1 \cdots d_r, d_{j_1} \cdots d_{j_s}). \]
Proof. As \(d_1, \ldots, d_r\) are discriminants such that no nonempty product \(d_{j_1} \cdots d_{j_s}\) \((1 \leq j_1 < \cdots < j_s \leq r)\) is a perfect square, we deduce that \(d_{j_1} \cdots d_{j_s}\) \((1 \leq j_1 < \cdots < j_s \leq r)\) is a discriminant. We have

\[
\sum_{n=a_1, \ldots, a_k \pmod{m}}^{\infty} \frac{1}{n^{2k}} = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} 
\]

The theorem now follows on appealing to Propositions 4.3 and 4.6.

\[\square\]

5 Examples

In this section we give some special cases of Theorem 4.7.

Theorem 5.1. Let \(k \in \mathbb{N}\). Then

\[
\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1} 2^{2k-2} \pi^{2k}}{2^{2k-1}(2k)!} A^* , \quad \sum_{n=2, 3 \pmod{5}}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1} 2^{2k-2} \pi^{2k}}{5^{2k+1}(2k)!} A^* ,
\]

where

\[A^* := 5(5^{2k} - 1)B_{2k} + \sum_{r=0}^{2k-1} \binom{2k}{r} 5^r (2^{2k-r} - 1)B_r \sqrt{5} .\]

Proof. The congruences \(n \equiv 1, 4 \pmod{5}\) are discriminantly determined as \(n \equiv 1, 4 \pmod{5} \Leftrightarrow (\frac{5}{n}) = +1\). By Theorem 4.7 we obtain

\[
\sum_{n=1, 4 \pmod{5}}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1} 2^{2k-2} B_{2k} \pi^{2k}}{(2k)!} P_k(5) + \frac{(-1)^{k-1} 2^{2k-2} \pi^{2k}}{(2k)!} \frac{H_k(5)}{\sqrt{5}} P_k(5, 5).
\]

By Definition 3.2 we have \(P_k(5) = \frac{5^{2k}-1}{5^{2k}}\) and \(P_k(1, 5) = P_k(5, 5) = 1\). By Definition 3.4 we have

\[H_k(5) = \frac{2}{5^{2k}} \sum_{r=0}^{2k-1} \binom{2k}{r} 5^r (1 - 2^{2k-r}) B_r .\]

The first asserted formula now follows. The second formula follows in a similar manner. \[\square\]

Taking \(k = 1\) and \(k = 2\) in Theorem 5.1, we obtain the following corollary.
Corollary 5.2. The following four evaluations hold:
\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2(5 + \sqrt{5})\pi^2}{125}, \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2(5 - \sqrt{5})\pi^2}{125},
\]
and
\[
\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{4(13 + 5\sqrt{5})\pi^4}{9375}, \quad \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{4(13 - 5\sqrt{5})\pi^4}{9375}.
\]

Theorem 5.3. Let \( k \in \mathbb{N} \). Then
\[
\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1}\pi^{2k}}{2^{4k+3}(2k)!} B^k,
\]
where
\[
B^k := 2^{4k+1}(2^{2k} - 1)B_{2k} + \sum_{r=0}^{2k-1} \binom{2k}{r} 2^{3(r-2^{k-r-1})} B_r \sqrt{2}.
\]

Proof. The congruences \( n \equiv 1, 7 \pmod{8} \) are discriminantly determined as \( n \equiv 1, 7 \pmod{8} \) \( \Rightarrow \left( \frac{6}{n} \right) = +1 \). By Theorem 4.7 with \( r = 1, d_1 = 8 \) and \( \epsilon_1 = +1 \), we obtain the first formula. For the second formula we choose \( r = 1, d_1 = 8 \) and \( \epsilon_1 = -1 \). \( \square \)

Taking \( k = 1 \) and \( k = 2 \) in Theorem 5.3, we obtain the following corollary.

Corollary 5.4. We have
\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{(2 + \sqrt{2})\pi^2}{32}, \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{(2 - \sqrt{2})\pi^2}{32},
\]
and
\[
\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{(16 + 11\sqrt{2})\pi^4}{3072}, \quad \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{(16 - 11\sqrt{2})\pi^4}{3072}.
\]

Theorem 5.5. Let \( k \in \mathbb{N} \). Then
\[
\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1}n^{2k}}{2^{2k+3}3^{2k+1}(2k)!} C^k,
\]
where
\[
C^k := 2^{2k}3(2^{2k} - 1)(3^{2k} - 1)B_{2k} + \sum_{r=0}^{2k-1} \binom{2k}{r} 2^{3r}(5^{2k-r} - 1)B_r \sqrt{3}.
\]

Proof. The congruences \( n \equiv 1, 11 \pmod{12} \) are discriminantly determined as \( n \equiv 1, 11 \pmod{12} \) \( \Rightarrow \left( \frac{12}{n} \right) = +1 \). By Theorem 4.7 with \( r = 1, d_1 = 12 \) and \( \epsilon_1 = +1 \), we obtain the first formula. For the second formula we choose \( r = 1, d_1 = 12 \) and \( \epsilon_1 = -1 \). \( \square \)

Taking \( k = 1 \) and \( k = 2 \) in Theorem 5.5, we obtain the following corollary.

Corollary 5.6. The following four evaluations hold:
\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{(2 + \sqrt{3})\pi^2}{36}, \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{(2 - \sqrt{3})\pi^2}{36},
\]
and
\[
\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{(40 + 23\sqrt{3})\pi^4}{7776}, \quad \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{(40 - 23\sqrt{3})\pi^4}{7776}.
\]
Theorem 5.7. Let \( k \in \mathbb{N} \). Then

\[
\sum_{n=1,9}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1} \pi^{2k}}{4 \cdot 5^{2k+1} (2k)!} D^+, \quad \sum_{n=3,7}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1} \pi^{2k}}{4 \cdot 5^{2k+1} (2k)!} D^-, 
\]

where

\[ D^\pm := 5(2^k - 1)(5^k - 1)B_{2k} \pm 2(2^k + 1) \sum_{r=0}^{2k-1} \left( \frac{2k}{r} \right) (2^{2k-r} - 1) 5' ! \sqrt{3}. \]

Proof. As \( n \equiv 1, 9 \pmod{10} \) \( \Leftrightarrow \left( \frac{20}{n} \right) = +1 \) we choose \( r = 1, d_1 = 20 \) and \( c_1 = +1 \) in Theorem 4.7 to obtain the first formula. For the second formula we choose \( r = 1, d_1 = 20 \) and \( c_1 = -1 \). 

Taking \( k = 1 \) and \( k = 2 \) in Theorem 5.7, we obtain the following corollary.

Corollary 5.8. The following four evaluations hold:

\[
\sum_{n=1,9}^{\infty} \frac{1}{n^2} = \frac{(3 + \sqrt{3}) \pi^2}{50}, \quad \sum_{n=3,7}^{\infty} \frac{1}{n^2} = \frac{(3 - \sqrt{3}) \pi^2}{50},
\]

and

\[
\sum_{n=1,9}^{\infty} \frac{1}{n^4} = \frac{(39 + 17 \sqrt{3}) \pi^4}{7500}, \quad \sum_{n=3,7}^{\infty} \frac{1}{n^4} = \frac{(39 - 17 \sqrt{3}) \pi^4}{7500}.
\]

Theorem 5.9. Let \( k \in \mathbb{N} \). Then

\[
\sum_{n=1,23}^{\infty} \frac{1}{n^{2k}} = E + F \sqrt{2} + G \sqrt{3} + H \sqrt{6}, 
\]

\[
\sum_{n=5,19}^{\infty} \frac{1}{n^{2k}} = E - F \sqrt{2} - G \sqrt{3} + H \sqrt{6}, 
\]

\[
\sum_{n=7,17}^{\infty} \frac{1}{n^{2k}} = E + F \sqrt{2} - G \sqrt{3} - H \sqrt{6}, 
\]

\[
\sum_{n=11,13}^{\infty} \frac{1}{n^{2k}} = E - F \sqrt{2} + G \sqrt{3} - H \sqrt{6},
\]

where

\[
E := \frac{(-1)^{k-1}(2^k - 1)(3^{2k} - 1) B_{2k} \pi^{2k}}{8 \cdot 3^{2k} (2k)!}, 
\]

\[
F := \frac{(-1)^{k-1}(3^{2k} + 1) \pi^{2k}}{2^{2k+3} \pi^{3k+1} (2k)!} \sum_{r=0}^{2k-1} \left( \frac{2k}{r} \right) 2^{3r} (1 - 3^{2k-r}) B_{r}, 
\]

\[
G := \frac{(-1)^{k-1} \pi^{2k}}{2^{2k+3} \pi^{3k+1} (2k)!} \sum_{r=0}^{2k-1} \left( \frac{2k}{r} \right) 2^{2r} 3^{3r} (1 - 5^{2k-r}) B_{r}, 
\]

\[
H := \frac{(-1)^{k-1} \pi^{2k}}{2^{2k+3} \pi^{3k+1} (2k)!} \sum_{r=0}^{2k-1} \left( \frac{2k}{r} \right) 2^{3r} 3^{3r} (1 + 5^{2k-r} - 7^{2k-r} - 11^{2k-r}) B_{r}, 
\]

Proof. We have

\[
\begin{align*}
\text{if } n \equiv 1, 23 \pmod{24} & \quad \Longleftrightarrow \quad \left( \frac{8}{n} \right) = \left( \frac{12}{n} \right) = 1, \\
\text{if } n \equiv 5, 19 \pmod{24} & \quad \Longleftrightarrow \quad \left( \frac{8}{n} \right) = \left( \frac{12}{n} \right) = -1, 
\end{align*}
\]
\[ n \equiv 7, 17 \pmod{24} \iff \left( \frac{8}{n} \right) = 1, \quad \left( \frac{12}{n} \right) = -1, \]
\[ n \equiv 11, 13 \pmod{24} \iff \left( \frac{8}{n} \right) = -1, \quad \left( \frac{12}{n} \right) = 1. \]

The asserted formulae follow by taking
\[
\begin{align*}
  r &= 2, \quad d_1 = 8, \quad d_2 = 12, \quad \epsilon_1 = 1, \quad \epsilon_2 = 1, \\
  r &= 2, \quad d_1 = 8, \quad d_2 = 12, \quad \epsilon_1 = -1, \quad \epsilon_2 = -1, \\
  r &= 2, \quad d_1 = 8, \quad d_2 = 12, \quad \epsilon_1 = 1, \quad \epsilon_2 = -1, \\
  r &= 2, \quad d_1 = 8, \quad d_2 = 12, \quad \epsilon_1 = -1, \quad \epsilon_2 = 1,
\end{align*}
\]
respectively, in Theorem 4.7.

Taking \( k = 1 \) in Theorem 5.9, we obtain the following corollary.

**Corollary 5.10.** The following four evaluations hold:

\[
\begin{align*}
  &\sum_{n=1, 23}^{\infty} \frac{1}{n^2} = \frac{(8 + 5\sqrt{2} + 4\sqrt{3} + 3\sqrt{6})n^2}{288}, \\
  &\sum_{n=5, 19}^{\infty} \frac{1}{n^2} = \frac{(8 - 5\sqrt{2} - 4\sqrt{3} + 3\sqrt{6})n^2}{288}, \\
  &\sum_{n=7, 17}^{\infty} \frac{1}{n^2} = \frac{(8 + 5\sqrt{2} - 4\sqrt{3} - 3\sqrt{6})n^2}{288}, \\
  &\sum_{n=11, 13}^{\infty} \frac{1}{n^2} = \frac{(8 - 5\sqrt{2} + 4\sqrt{3} - 3\sqrt{6})n^2}{288}.
\end{align*}
\]

**Theorem 5.11.** Let \( k \in \mathbb{N} \). Then

\[
\sum_{n=1, 3, 9, 19, 25, 27}^{\infty} \frac{1}{n^{2k}} = J + K\sqrt{7}, \quad \sum_{n=5, 11, 13, 15, 17, 23}^{\infty} \frac{1}{n^{2k}} = J - K\sqrt{7},
\]

where

\[
J := \frac{(-1)^{k-1}(2^{2k-1} - 1)(7^{2k} - 1)B_{2k}n^{2k}}{2^{2k}7^{2k}(2k)!},
\]

\[
K := \frac{(-1)^{k-1}n^{2k}}{2^{2k}7^{2k+1}(2k)!} \sum_{r=0}^{2k-1} \binom{2k}{r} 2^{2r}7^r(1^{2k-r} + 3^{2k-r} - 5^{2k-r} + 9^{2k-r} - 11^{2k-r} - 13^{2k-r})B_r.
\]

**Proof.** We have
\[
n \equiv 1, 3, 9, 19, 25, 27 \pmod{28} \iff \left( \frac{28}{n} \right) = 1
\]
and
\[
n \equiv 5, 11, 13, 15, 17, 23 \pmod{28} \iff \left( \frac{28}{n} \right) = -1
\]
so the congruences are discriminantly determined and we can apply Theorem 4.7 with \( d = \Delta = 28 \) and \( f = 1 \).

Taking \( k = 1 \) in Theorem 5.11 we obtain the following result.

**Corollary 5.12.** The following two evaluations hold:

\[
\sum_{n=1, 3, 9, 19, 25, 27}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{49}(3 + \sqrt{7}), \quad \sum_{n=5, 11, 13, 15, 17, 23}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{49}(3 - \sqrt{7}).
\]
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References