Bounds for the Representations of Integers by Positive Quadratic Forms

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The fact that every positive integer is the sum of four squares of integers was first proved in 1770 by Lagrange [**9**]. For example the integer 1770 is $1^2 + 1^2 + 2^2 + 42^2$. We can view Lagrange's theorem as telling us that the polynomial $x_1^2 + x_2^2 + x_3^2 + x_4^2$ in the four real variables x_1, x_2, x_3 and x_4 represents every positive integer since whatever positive integer n is specified there are integer values of x_1, x_2, x_3 and x_4, say y_1, y_2, y_3, and y_4, respectively, such that $y_1^2 + y_2^2 + y_3^2 + y_4^2 = n$. The quadruple (y_1, y_2, y_3, y_4) is called the representation of n by the polynomial $x_1^2 + x_2^2 + x_3^2 + x_4^2$. The polynomial $x_1^2 + x_2^2 + x_3^2 + x_4^2$ is an example of a quadratic form as it has the property that if we replace each of x_1, x_2, x_3, x_4 by tx_1, tx_2, tx_3, tx_4, respectively, where t is a real number, we obtain t^2 times the original polynomial. (All quadratic forms in this article are assumed to have integer coefficients.) Moreover, the quadratic form $x_1^2 + x_2^2 + x_3^2 + x_4^2$ has the additional property that it is nonnegative for all real values of x_1, x_2, x_3, and x_4 and is zero only when x_1, x_2, x_3, and x_4 are all zero. Quadratic forms with this property are called positive. (Frobenius gave in 1894 necessary and sufficient conditions for a quadratic form to be positive, see for example [**11**, Theorem 13.3.1, p. 400].) A positive quadratic form is called universal if it represents every positive integer. Thus $x_1^2 + x_2^2 + x_3^2 + x_4^2$ is an example of a universal positive quadratic form.

The problem of determining whether a given positive quadratic form is universal has been solved in some ground-breaking work in recent years. It was first solved by Conway and Schneeberger [**5**] for positive quadratic forms all of whose cross-product terms have even coefficients. (For example the cross-product terms of the positive quadratic form $3x_1^2 + 3x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_1x_3$ are all even, namely 2, 2, and 0.) They proved but did not publish the proof of the following theorem.

15-Theorem. *Let f be a positive quadratic form in any number of variables such that all the coefficients of its cross-product terms are even integers. If f represents all the positive integers up to and including* 15 *then f is universal.*

The details of their work are given in [**5**] and [**13**]. In 2000 Bhargava [**2**] gave a beautiful new proof of this theorem in the following stronger form.

Strong 15-Theorem. *Let f be a positive quadratic form in any number of variables such that all the coefficients of its cross-product terms are even integers. If f represents all the nine integers*

$$1, 2, 3, 5, 6, 7, 10, 14, 15$$

then it is universal.

Bhargava also showed that the set $\{1, 2, 3, 5, 6, 7, 10, 14, 15\}$ is minimal in a certain sense.

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In 1993 Conway formulated the conjecture that a positive quadratic form that represents all the positive integers up to and including 290 must be universal. This was proved by Bhargava and Hanke [**3**].

290-Theorem. *If a positive quadratic form in any number of variables represents all the positive integers up to and including* 290 *then it is universal.*

Indeed Bhargava and Hanke proved this result in the following stronger form.

Strong 290-Theorem. *If a positive quadratic form in any number of variables represents all the twenty-nine integers*

$$1, 2, 3, 5, 6, 7, 10, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29,$$

$$30, 31, 34, 35, 37, 42, 58, 93, 110, 145, 203, 290$$

then it is universal.

Bhargava and Hanke showed too that the set in the strong 290-theorem is also minimal.

The beauty of these results is that given a positive quadratic form we have only to check that it represents nine integers to know that it represents all positive integers in the "all cross-product coefficients even" case and twenty-nine integers in the case that the form has at least one odd cross-product coefficient. This leads us naturally to the question "How do we check if a positive quadratic form represents a certain positive integer?" The main result of this article provides a simple answer to this question. Let us consider an example for guidance. We show for the quadratic form considered in this example that for each positive integer n there are only finitely many representations of n by the form and moreover that all such representations lie in a certain hypercube.

Example. We consider the quadratic form

$$\begin{aligned} G(x_1, x_2, x_3, x_4) := 3x_1^2 + 44x_2^2 + 13x_3^2 + 18x_4^2 + 2x_1x_2 + 6x_1x_3 + 8x_1x_4 \\ + 42x_2x_3 + 16x_2x_4 + 8x_3x_4. \end{aligned} \tag{1}$$

We note that in this example all the cross-product terms of G have even coefficients. We leave it to the reader to check that G satisfies the determinantal conditions of Frobenius so that G is a positive form. Alternatively we can see that G is a positive form from the identity

$$\begin{aligned} 4323G = 1441(3x_1 + x_2 + 3x_3 + 4x_4)^2 + 11(131x_2 + 60x_3 + 20x_4)^2 \\ + 30(11x_3 - 40x_4)^2 + 2358x_4^2, \end{aligned}$$

which was found by first completing the square in x_1 in G, then the square in x_2 and finally the square in x_3. (The coefficients of the squares in this identity are all positive and G can only be zero when

$$3x_1 + x_2 + 3x_3 + 4x_4 = 131x_2 + 60x_3 + 20x_4 = 11x_3 - 40x_4 = x_4 = 0,$$

that is when $x_1 = x_2 = x_3 = x_4 = 0$, so G is positive.)

Our idea is to attempt to express G in the form

$$\begin{aligned}G(x_1, x_2, x_3, x_4) &= G(0, x_2 - tx_1, x_3 - ux_1, x_4 - vx_1) + wx_1^2\\ &= 44(x_2 - tx_1)^2 + 13(x_3 - ux_1)^2 + 18(x_4 - vx_1)^2\\ &\quad + 42(x_2 - tx_1)(x_3 - ux_1) + 16(x_2 - tx_1)(x_4 - vx_1)\\ &\quad + 8(x_3 - ux_1)(x_4 - vx_1) + wx_1^2\end{aligned}$$

for some rational numbers t, u, v, and w. Clearly the coefficients of x_2^2, x_3^2, x_4^2, x_2x_3, x_2x_4, and x_3x_4 agree so we have only to arrange the agreement of the coefficients of x_1^2, x_1x_2, x_1x_3, and x_1x_4. Equating the coefficients of x_1x_2, x_1x_3, and x_1x_4, we obtain

$$\begin{aligned}2 &= -88t - 42u - 16v,\\ 6 &= -42t - 26u - 8v,\\ 8 &= -16t - 8u - 36v.\end{aligned}$$

Solving these linear equations for t, u, and v we obtain

$$t = \frac{150}{361}, \quad u = -\frac{301}{361}, \quad v = -\frac{80}{361}.$$

Equating the coefficients of x_1^2, we have

$$3 = 44t^2 + 13u^2 + 18v^2 + 42tu + 16tv + 8uv + w.$$

Using the values of t, u, and v in this equation, we deduce $w = \frac{10}{361}$. This shows that

$$\begin{aligned}G = {}& 44\left(x_2 - \frac{150}{361}x_1\right)^2 + 13\left(x_3 + \frac{301}{361}x_1\right)^2 + 18\left(x_4 + \frac{80}{361}x_1\right)^2\\ &+ 42\left(x_2 - \frac{150}{361}x_1\right)\left(x_3 + \frac{301}{361}x_1\right) + 16\left(x_2 - \frac{150}{361}x_1\right)\left(x_4 + \frac{80}{361}x_1\right)\\ &+ 8\left(x_3 + \frac{301}{361}x_1\right)\left(x_4 + \frac{80}{361}x_1\right) + \frac{10}{361}x_1^2.\end{aligned}$$

Thus, if n is a positive integer which is represented by G, then there are integers y_1, y_2, y_3, and y_4 such that $G(y_1, y_2, y_3, y_4) = n$ and our expression for G gives

$$n = G(0, y_2 - \frac{150}{361}y_1, y_3 + \frac{301}{361}y_1, y_4 + \frac{80}{361}y_1) + \frac{10}{361}y_1^2.$$

As $G(x_1, x_2, x_3, x_4)$ is a positive quadratic form so is $G(0, x_2, x_3, x_4)$ and we have

$$G(0, y_2 - \frac{150}{361}y_1, y_3 + \frac{301}{361}y_1, y_4 + \frac{80}{361}y_1) \geq 0$$

and thus

$$n \geq \frac{10}{361}y_1^2.$$

This shows that for a fixed positive integer n there are only finitely many possibilities for the integer y_1 given by

$$|y_1| \leq \sqrt{\frac{361}{10}n}.$$

Similarly we obtain

$$
\begin{aligned}
G(x_1, x_2, x_3, x_4) &= G\left(x_1 - \frac{45}{19}x_2, 0, x_3 + 2x_2, x_4 + \frac{10}{19}x_2\right) + \frac{3}{19}x_2^2, \\
&= G\left(x_1 + \frac{129}{109}x_3, x_2 + \frac{380}{763}x_3, 0, x_4 - \frac{200}{763}x_2\right) + \frac{30}{763}x_3^2, \\
&= G\left(x_1 + \frac{48}{11}x_4, x_2 + \frac{20}{11}x_4, x_3 - \frac{40}{11}x_4\right) + \frac{6}{11}x_4^2,
\end{aligned}
$$

so that

$$
|y_2| \le \sqrt{\frac{19}{3}n}, \quad |y_3| \le \sqrt{\frac{763}{30}n}, \quad |y_4| \le \sqrt{\frac{11}{6}n}.
$$

In this example we have shown that there are only finitely many solutions in integers y_1, y_2, y_3, y_4 to $G(y_1, y_2, y_3, y_4) = n$ for a given positive integer n, and every such solution vector (y_1, y_2, y_3, y_4) lies in the hypercube

$$
|y_1| \le \sqrt{\frac{361}{10}n}, \quad |y_2| \le \sqrt{\frac{19}{3}n}, \quad |y_3| \le \sqrt{\frac{763}{30}n}, \quad |y_4| \le \sqrt{\frac{11}{6}n}.
$$

The main result of this article is the explicit determination of the corresponding hypercube for a general positive quadratic form. Thus, in view of the 15- and 290-Theorems, we have only to check the finite number of values of a given positive quadratic form on the integral points of a hypercube to determine its universality or nonuniversality. Before proving this result we make a few remarks about universal and nonuniversal positive quadratic forms.

Universal and nonuniversal forms

Clearly a positive quadratic form in one variable cannot be universal. Such a form is $a_1x_1^2$, where a_1 is a positive integer, and $a_1x_1^2$ cannot represent the positive integer $2a_1$. What about positive quadratic forms in two variables? Such forms in two variables are called binary. The general binary quadratic form is $a_1x_1^2 + a_{12}x_1x_2 + a_2x_2^2$, where a_1, a_{12} and a_2 are integers. As

$$
a_1x_1^2 + a_{12}x_1x_2 + a_2x_2^2 = \frac{1}{4a_1}(2a_1x_1 + a_{12}x_2)^2 + \frac{(4a_1a_2 - a_{12}^2)}{4a_1}x_2^2,
$$

we see that the form is positive if and only if

$$
a_1 > 0, \quad 4a_1a_2 - a_{12}^2 > 0.
$$

Proposition. *A positive binary quadratic form $a_1x_1^2 + a_{12}x_1x_2 + a_2x_2^2$ is never universal.*

Proof. The integer $d := a_{12}^2 - 4a_1a_2$ is strictly negative so it is not a perfect square. Thus, from the theory of quadratic residues modulo a prime, we know that there are infinitely many primes p such that d is a quadratic nonresidue modulo p. Hence we can choose the prime p to satisfy $p > 4a_1|d|$. Suppose that $a_1x_1^2 + a_{12}x_1x_2 + a_2x_2^2$ is universal. Then it represents p and so there are integers u and v such that

$$
p = a_1u^2 + a_{12}uv + a_2v^2.
$$

Hence

$$4a_1 p = x^2 - dy^2, \text{ where } x := 2a_1 u + a_{12} v \text{ and } y := v.$$

If p divides y then, from $4a_1 p = x^2 - dy^2$, we see that p divides x. Thus p^2 divides $x^2 - dy^2 = 4a_1 p$. This is impossible as $p > 4a_1$. Hence p does not divide y. Thus there is an integer z such that $yz \equiv 1 \pmod{p}$. Then $(xz)^2 \equiv d \pmod{p}$, contradicting that d is a quadratic nonresidue modulo p. Therefore, $a_1 x_1^2 + a_{12} x_1 x_2 + a_2 x_2^2$ cannot be universal. ■

What about a positive quadratic form in three variables? Such forms are called ternary quadratic forms and they like binary quadratic forms can never be universal. To prove that a general positive ternary quadratic form is never universal is much more difficult than in the binary case. Proofs can be found in Albert [**1**, p. 291, Theorem 13] and Conway [**4**, p. 142]. In the diagonal case, which is when there are no cross-product terms in the ternary form (so that the ternary form is $a_1 x_1^2 + a_2 x^2 + a_3 x_3^2$), a simple proof has been given by Dickson [**8**, p. 104, Theorem 95].

What about positive quadratic forms in four variables? These forms are called quaternary. Can they represent all positive integers? Since $a^2 + b^2 \equiv 0, 1, 2 \pmod{4}$ for any integers a and b, the form $x_1^2 + x_2^2 + 4a_3 x_3^2 + 4a_4 x_4^2$, where a_3 and a_4 are positive integers, cannot represent any positive integer $n \equiv 3 \pmod{4}$. Thus there are infinitely many positive quaternary quadratic forms which are not universal. On the other hand, Liouville and other mathematicians showed that there are positive quaternary quadratic forms different from $x_1^2 + x_2^2 + x_3^2 + x_4^2$, which are universal. An example of Liouville [**10**, p. 271] follows easily from Lagrange's theorem. Let n be a positive integer and let a, b, c, and d be integers such that $n = a^2 + b^2 + c^2 + d^2$. Since there are only two possible residues for an integer modulo 2, namely 0 and 1, by Dirichlet's box principle at least two of b, c, and d must have the same residue modulo 2, say $c \equiv d \pmod{2}$. Then $n = a^2 + b^2 + 2e^2 + 2f^2$, where e and f are the integers $(c+d)/2$ and $(c-d)/2$, respectively. Thus the quadratic form $x_1^2 + x_2^2 + 2x_3^2 + 2x_4^2$ is universal. Ramanujan [**12**] and Dickson [**6, 8**] determined all the universal positive quaternary quadratic forms $a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + a_4 x_4^2$, where a_1, a_2, a_3, and a_4 are positive integers satisfying $a_1 \le a_2 \le a_3 \le a_4$. They proved that there are precisely 54 such forms. The interested reader can find them listed in Dickson [**8**, p. 105]. Many examples of universal positive quaternary quadratic forms with nonzero cross-product terms have been given by Dickson, see Dickson [**7**].

For forms in more than four variables there are infinitely many such forms which are universal, for example $x_1^2 + x_2^2 + x_3^2 + x_4^2 + a_5 x_5^2 + \cdots + a_n x_n^2$, where $a_5, \ldots, a_n$ are positive integers, and infinitely many forms which are not, for example $x_1^2 + x_2^2 + 4a_3 x_3^2 + \cdots + 4a_n x_n^2$, where $a_3, \ldots, a_n$ are positive integers.

These examples show the wide variety of possibilities for the universality or nonuniversality of positive quadratic forms and so demonstrate the simplicity and power of the 15- and 290-Theorems.

Notation and Main Result

We now state and prove the central result of this article, which we have not found in the literature. We denote the sets of integers, positive integers and rational numbers by $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{Q}$, respectively. The general quadratic form with integer coefficients in k real variables $x_1, \ldots, x_k$ is given by

$$F(x_1, \ldots, x_k) := \sum_{1 \le i \le j \le k} a_{ij} x_i x_j, \quad a_{ij} \in \mathbb{Z}. \tag{2}$$

We assume that F is positive so that the coefficients a_{ij} satisfy the previously mentioned conditions of Frobenius. Let n be a positive integer. Our objective is to give an explicit hypercube in $\mathbb{Z}^k$ in which all the solutions in integers of $F(x_1, \ldots, x_k) = n$ (if any) must lie.

The matrix of the form F is the $k \times k$ symmetric matrix

$$A := \begin{bmatrix} a_{11} & \frac{1}{2}a_{12} & \cdots & \frac{1}{2}a_{1k} \\ \frac{1}{2}a_{12} & a_{22} & \cdots & \frac{1}{2}a_{2k} \\ \cdot & \cdot & \cdots & \cdot \\ \frac{1}{2}a_{1k} & \frac{1}{2}a_{2k} & \cdots & a_{kk} \end{bmatrix},$$

where each entry of A is either an integer or half an odd integer. Thus for the quadratic form G in the example (see (1)) we have

$$A = \begin{bmatrix} 3 & 1 & 3 & 4 \\ 1 & 44 & 21 & 8 \\ 3 & 21 & 13 & 4 \\ 4 & 8 & 4 & 18 \end{bmatrix}.$$

If all the coefficients of the cross-product terms in F are even, then all the entries in A are integers and the form F is said to be an integer-matrix form. The form F and its matrix A are related by $F = X^t A X$, where ^{t} denotes the transpose of a matrix and $X = [x_1 \cdots x_k]^t$. As F is a positive quadratic form, and A is the matrix of F, by the determinantal conditions of Frobenius we have $\det A > 0$. In the case of G we have $\det A = 60$.

For $j = 1, 2, \ldots, k$ with $k \geq 2$ we let A_j denote the $(k-1) \times (k-1)$ symmetric matrix formed by deleting the jth row and jth column of A. The principal diagonal of A_j $(j = 1, 2, \ldots, k)$ is part of the principal diagonal of A so, as F is a positive quadratic form with matrix A, again by the Frobenius determinantal conditions for a quadratic form to be positive, we have $\det A_j > 0$ $(j = 1, 2, \ldots, k)$. In the case of G we have

$$A_1 = \begin{bmatrix} 44 & 21 & 8 \\ 21 & 13 & 4 \\ 8 & 4 & 18 \end{bmatrix}, A_2 = \begin{bmatrix} 3 & 3 & 4 \\ 3 & 13 & 4 \\ 4 & 4 & 18 \end{bmatrix}, A_3 = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 44 & 8 \\ 4 & 8 & 18 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 3 & 1 & 3 \\ 1 & 44 & 21 \\ 3 & 21 & 13 \end{bmatrix},$$

and

$$\det A_1 = 2166, \ \det A_2 = 380, \ \det A_3 = 1526, \ \det A_4 = 110.$$

We prove our main result by using some elementary computational matrix algebra to express $F(x_1, \ldots, x_k)$ for $r = 1, \ldots, k$ in the form

$$F(x_1 - t_1 x_r, \ldots, x_{r-1} - t_{r-1} x_r, 0, x_{r+1} - t_{r+1} x_r, \ldots, x_k - t_k x_r) + u_r x_r^2$$

for some rational numbers $t_1, \ldots, t_{r-1}, t_{r+1}, \ldots, t_k$ and an explicitly known rational number u_r as suggested by the example.

Theorem. *Let $k \in \mathbb{N}$ be such that $k \geq 2$. Let F be the integral positive quadratic form in the k indeterminates $x_1, \ldots, x_k$ given in* (2). *Let $n \in \mathbb{N}$. Then there are only finitely* *many solutions* $(y_1, \ldots, y_k) \in \mathbb{Z}^k$ *of*

$$F(y_1, \ldots, y_k) = n$$

and any such solution satisfies

$$|y_r| \le \left\lfloor \sqrt{\frac{\det A_r}{\det A}} \sqrt{n} \right\rfloor, \quad r = 1, 2, \ldots, k,$$

where $\lfloor x \rfloor$ *denotes the floor of the real number* x.

Proof. Fix $r \in \{1, 2, \ldots, k\}$. As $\det A_r \neq 0$ the system of $k-1$ linear equations in the $k-1$ unknowns $t_1, \ldots, t_{r-1}, t_{r+1}, \ldots, t_k$

$$A_r \begin{bmatrix} t_1 \\ \vdots \\ t_{r-1} \\ t_{r+1} \\ \vdots \\ t_k \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}a_{1\,r} \\ \vdots \\ -\frac{1}{2}a_{r-1\,r} \\ -\frac{1}{2}a_{r\,r+1} \\ \vdots \\ -\frac{1}{2}a_{r\,k} \end{bmatrix} \tag{3}$$

has a unique solution $(t_1, \ldots, t_{r-1}, t_{r+1}, \ldots, t_k) \in \mathbb{Q}^{k-1}$. Let T be the $k \times k$ matrix defined by

$$T := \begin{bmatrix} 1 & 0 & \cdots & 0 & t_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & t_{r-1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & t_{r+1} & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \ldots & 0 & t_k & 0 & \cdots & 1 \end{bmatrix},$$

where the column with first entry t_1 is the rth column. Expanding the determinant of T by the rth row we obtain $\det T = 1$.

Suppose now that $(x_1, \ldots, x_k) \in \mathbb{Z}^k$ is a solution of $F(x_1, \ldots, x_k) = n$. For $i = 1, 2, \ldots, k$ define $y_i \in \mathbb{Q}$ by

$$y_i := \begin{cases} x_i - t_i x_r & \text{if } i \neq r, \\ x_r & \text{if } i = r. \end{cases} \tag{4}$$

Let

$$X := \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}, \quad Y := \begin{bmatrix} y_1 \\ \vdots \\ y_k \end{bmatrix},$$

so that $n = X^t A X$ and $X = TY$. Hence

$$n = (TY)^t A (TY) = Y^t (T^t A T) Y. \tag{5}$$

We now determine the matrix T^tAT. The matrix T^tA is just the matrix A with the rth row replaced by $[u_1\, u_2 \cdots u_k]$, where for $i = 1, 2, \ldots, k$

$$u_i = \begin{bmatrix} t_1 & \cdots & t_{r-1} & 1 & t_{r+1} & \cdots & t_k \end{bmatrix} \begin{bmatrix} \tfrac{1}{2}a_{1\,i} & \cdots & \tfrac{1}{2}a_{i-1\,i} & a_{i\,i} & \tfrac{1}{2}a_{i\,i+1} & \cdots & \tfrac{1}{2}a_{i\,k} \end{bmatrix}^t.$$

If $i = r$ we have

$$u_r = \sum_{j=1}^{r-1} \frac{1}{2} a_{j\,r} t_j + a_{r\,r} + \sum_{j=r+1}^{k} \frac{1}{2} a_{r\,j} t_j.$$

If $i < r$ we have

$$u_i = \sum_{j=1}^{i-1} \frac{1}{2} a_{j\,i} t_j + a_{i\,i} t_i + \sum_{j=i+1}^{r-1} \frac{1}{2} a_{i\,j} t_j + \frac{1}{2} a_{i\,r} + \sum_{j=r+1}^{k} \frac{1}{2} a_{i\,j} t_j = 0,$$

appealing to (3). Similarly if $i > r$ we have

$$u_i = \sum_{j=1}^{r-1} \frac{1}{2} a_{j\,i} t_j + \frac{1}{2} a_{r\,i} + \sum_{j=r+1}^{i-1} \frac{1}{2} a_{j\,i} t_j + a_{i\,i} t_i + \sum_{j=i+1}^{k} \frac{1}{2} a_{i\,j} t_j = 0,$$

again appealing to (3). Thus the rth row of T^tA is $\begin{bmatrix} 0 & \cdots & 0 & u_r & 0 & \cdots & 0 \end{bmatrix}$. Hence the matrix T^tAT is the same as the matrix A except that the rth row is $[0 \cdots 0\ u_r\ 0 \cdots 0]$ and the rth column is $[0 \cdots 0\ u_r\ 0 \cdots 0]^t$. Thus by (5) we have

$$\begin{aligned} n = Y^t(T^tAT)Y &= \sum_{\substack{1 \le i \le j \le k \\ i,\, j \ne r}} a_{i\,j} y_i y_j + u_r y_r^2 \\ &= F(y_1, \ldots, y_{r-1}, 0, y_{r+1}, \ldots, y_k) + u_r y_r^2. \end{aligned} \tag{6}$$

As F is a positive form, we have $F(y_1, \ldots, y_{r-1}, 0, y_{r+1}, \ldots, y_k) \ge 0$, and thus from (4) and (6), we deduce

$$n \ge u_r x_r^2. \tag{7}$$

All that remains is to determine u_r. Expanding $\det(T^tAT)$ by the rth row, we deduce that

$$\det(T^tAT) = u_r \det A_r.$$

As $\det T = 1$ we have

$$\det(T^tAT) = \det(T^t) \det A \det T = \det A.$$

Hence

$$u_r = \frac{\det A}{\det A_r}. \tag{8}$$

Finally, from (7) and (8), we obtain

$$|x_r| \le \sqrt{\frac{\det A_r}{\det A}} \sqrt{n}, \qquad r = 1, 2, \ldots, k,$$

from which the theorem follows. ∎

We remark that the theorem is best possible in the sense that the hypercube cannot in general be made smaller and still contain all the solutions $(y_1, \dots, y_k) \in \mathbb{Z}^k$ of $F(y_1, \dots, y_k) = n$. To see this take for example $F(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$ (so that $k = 3$) and $n = 3$. Here

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_1 = A_2 = A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

so that $\det A = \det A_1 = \det A_2 = \det A_3 = 1$ and as $\lfloor \sqrt{3} \rfloor = 1$ the hypercube is

$$|y_r| \leq 1, \quad r = 1, 2, 3.$$

The only solutions of $y_1^2 + y_2^2 + y_3^2 = 3$ are $(y_1, y_2, y_3) = (\pm 1, \pm 1, \pm 1)$ (8 choices of sign) and all of these lie on the boundary of the hypercube.

Example (continued). We apply our theorem in conjunction with the strong 15-Theorem to show that G (defined in (1)) is universal. By the theorem any solution $(y_1, y_2, y_3, y_4) \in \mathbb{Z}^4$ of $G(y_1, y_2, y_3, y_4) = n$ must satisfy

$$|y_1| \leq \left\lfloor \sqrt{\frac{\det A_1}{\det A}} \sqrt{n} \right\rfloor = \left\lfloor \sqrt{\frac{361}{10}} \sqrt{n} \right\rfloor,$$

$$|y_2| \leq \left\lfloor \sqrt{\frac{\det A_2}{\det A}} \sqrt{n} \right\rfloor = \left\lfloor \sqrt{\frac{19}{3}} \sqrt{n} \right\rfloor,$$

$$|y_3| \leq \left\lfloor \sqrt{\frac{\det A_3}{\det A}} \sqrt{n} \right\rfloor = \left\lfloor \sqrt{\frac{763}{30}} \sqrt{n} \right\rfloor,$$

$$|y_4| \leq \left\lfloor \sqrt{\frac{\det A_4}{\det A}} \sqrt{n} \right\rfloor = \left\lfloor \sqrt{\frac{11}{6}} \sqrt{n} \right\rfloor.$$

We remark that these are the same bounds that we obtained previously in the example. A simple computer search through these ranges for each $n \in \{1, 2, 3, 5, 6, 7, 10, 14, 15\}$ found a solution for each of the nine values of n.

n	(y_1, y_2, y_3, y_4)	n	(y_1, y_2, y_3, y_4)
1	$(-5, -2, 4, 1)$	7	$(-13, -5, 10, 3)$
2	$(-4, -2, 4, 1)$	10	$(-1, 0, 1, 0)$
3	$(-9, -4, 8, 2)$	14	$(-22, -9, 18, 5)$
5	$(-7, -3, 6, 2)$	15	$(-21, -9, 18, 5)$
6	$(-14, -6, 12, 3)$		

Thus, by the strong 15-Theorem, $G(x_1, x_2, x_3, x_4)$ is universal.

The solutions (y_1, y_2, y_3, y_4) in the table are not unique since if (y_1, y_2, y_3, y_4) is a solution so is $(-y_1, -y_2, -y_3, -y_4)$. However, if we identify the two solutions $\pm(y_1, y_2, y_3, y_4)$ the solutions given in the table for $n = 1$ and $n = 2$ are unique. For $n = 6$, with this identification, there are six solutions, namely,

$$\pm(2, 1, -2, -1), \quad \pm(4, 1, -2, -1), \quad \pm(6, 2, -4, -1),$$

$$\pm(6, 3, -6, -1), \quad \pm(12, 5, -10, -3), \quad \pm(14, 6, -12, -3).$$

The solution $\pm(14, 6, -12, -3)$ lies on the boundary of the hypercube as

$$\left\lfloor \sqrt{\frac{361}{10}\sqrt{6}} \right\rfloor = 14, \quad \left\lfloor \sqrt{\frac{19}{3}\sqrt{6}} \right\rfloor = 6, \quad \left\lfloor \sqrt{\frac{763}{30}\sqrt{6}} \right\rfloor = 12, \quad \left\lfloor \sqrt{\frac{11}{6}\sqrt{6}} \right\rfloor = 3,$$

again showing that in general the theorem is best possible.

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Summary. Recent ground-breaking work of Conway, Schneeberger, Bhargava, and Hanke shows that to determine whether a given positive quadratic form F with integer coefficients represents every positive integer (and so is universal), it is only necessary to check that F represents all the integers in an explicitly given finite set S of positive integers. The set contains either nine or twenty-nine integers depending on the parity of the coefficients of the cross-product terms in F and is otherwise independent of F. In this article we show that F represents a given positive integer n if and only if $F(y_1, \ldots, y_k) = n$ for some integers $y_1, \ldots, y_k$ satisfying $|y_i| \leq \sqrt{c_i n}$, $i = 1, \ldots, k$, where the positive rational numbers c_i are explicitly given and depend only on F. Let m be the largest integer in S (in fact $m = 15$ or 290). Putting these results together we have

$$F \text{ is universal if and only if } S \subseteq \{F(y_1, \ldots, y_k) \mid |y_i| \leq \sqrt{c_i m}, i = 1, \ldots, k\}.$$

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