A “Four Integers” Theorem and a “Five Integers” Theorem

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Abstract. The recent exciting results by Bhargava, Conway, Hanke, Kaplansky, Rouse, and Schneeberger concerning the representabiltity of integers by positive integral quadratic forms in any number of variables are presented. These results build on the earlier work of Dickson, Halmos, Ramanujan, and Willerding on quadratic forms. Two results of this type for positive diagonal ternary forms are proved. These are the “four integers” and “five integers” theorems of the title.

An integral quadratic form in the variables $x_1, \ldots, x_k$ is a homogeneous polynomial

$$f(x_1, \ldots, x_k) = \sum_{1 \leqslant i \leqslant j \leqslant k} a_{ij} x_i x_j,$$

where the a_{ij} are integers. Clearly $f(0, \ldots, 0) = 0$. If $k = 2$ the quadratic form is called binary, if $k = 3$ ternary, and if $k = 4$ quaternary. The matrix of the form f is the $k \times k$ symmetric matrix

$$F := \begin{pmatrix} a_{11} & \frac{1}{2}a_{12} & \ldots & \frac{1}{2}a_{1k} \\ \frac{1}{2}a_{12} & a_{22} & \ldots & \frac{1}{2}a_{2k} \\ \cdot & \cdot & \ldots & \cdot \\ \frac{1}{2}a_{1k} & \frac{1}{2}a_{2k} & \ldots & a_{kk} \end{pmatrix}.$$

The form f is said to be diagonal if the matrix F is a diagonal matrix; that is $a_{12} = a_{13} = \cdots = a_{k-1\,k} = 0$, so that $f = a_{11}x_1^2 + \cdots + a_{kk}x_k^2$. The form f is said to be an integer-matrix form if all the entries of F are integers; that is $a_{12}, a_{13}, \ldots, a_{k-1\,k}$ are all even integers.

If $f(x_1, \ldots, x_k) > 0$ for all integers $x_1, \ldots, x_k$ with $(x_1, \ldots, x_k) \neq (0, \ldots, 0)$ we say that the form f is positive. We consider only positive forms throughout this article. Frobenius gave a necessary and sufficient condition for a quadratic form to be positive in 1894, see for example [**18**, p. 400].

An integer n is said to be represented by f if there exist integers $y_1, \ldots, y_k$ such that $n = f(y_1, \ldots, y_k)$. The set of integers represented by a positive form f comprises a certain set of positive integers together with 0. If f represents every positive integer, then f is said to be universal. Two positive integral quadratic forms $f(x_1, \ldots, x_k)$ and $g(x_1, \ldots, x_k)$ with matrices F and G, respectively, are said to be equivalent if there exists a $k \times k$ matrix U with integral entries and $\det U = \pm 1$ such that $G = U^T F U$, where U^T denotes the transpose of the matrix U. The class $[f]$ of the form f is the set of forms g which are equivalent to f. As forms in the same class represent the same integers, we usually identify a form with its class when discussing representability and universality.

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It is a classical theorem due to Lagrange [**14**] that every positive integer is the sum of four integral squares; that is the positive diagonal quaternary integral quadratic form $x_1^2 + x_2^2 + x_3^2 + x_4^2$ represents every positive integer, and so is a universal form. Liouville [**16**] (and other mathematicians) determined further universal positive diagonal quaternary integral quadratic forms such as $x_1^2 + x_2^2 + 2x_3^2 + 2x_4^2$. On the other hand, the form $x_1^2 + x_2^2 + 5x_3^2 + 5x_4^2$ is not universal as it does not represent 3. Liouville [**15**] showed that it represents every positive integer except 3. Liouville [**17**] also showed that the form $x_1^2 + x_2^2 + 4x_3^2 + 4x_4^2$ represents every positive integer except the infinitely many integers $4m + 3$ ($m = 0, 1, 2, \ldots$). Ramanujan [**20**] and Dickson [**6**] determined all positive diagonal quaternary integral quadratic forms which are universal. There are precisely 54 of them. Ramanujan had claimed there were 55 such forms but Dickson noted that one of Ramanujan's forms was not universal. Willerding [**23, 24**] treated (classes of) positive integer-matrix quaternary quadratic forms which are universal and claimed that there are 178 (classes of) such forms. This count was later shown to be incorrect. As regards universality of positive integral quadratic forms, the case of four variables is a "threshold" since in more than four variables there are infinitely many universal forms. For example $x_1^2 + x_2^2 + x_3^2 + x_4^2 + mx_5^2$ is universal for any positive integer m (in this connection see Dickson [**5**]), whereas in fewer than four variables there are no universal forms. We can easily see this in the case of positive diagonal ternary quadratic forms $ax_1^2 + bx_2^2 + cx_3^2$, where a, b, and c are positive integers with $a \leqslant b \leqslant c$. This follows since to represent 1 we must have $a = 1$, to represent 2 we must have $b = 1$ or $b = 2$, and $x^2 + y^2 + cz^2$ does not represent 7 if $c = 1$; 14 if $c = 2$; 6 if $c = 3$, and 3 if $c \geqslant 4$, and $x^2 + 2y^2 + cz^2$ does not represent 7 if $c = 2$; 10 if $c = 3$; 14 if $c = 4$; 10 if $c = 5$; and 5 if $c \geqslant 6$. A proof that no positive ternary integral quadratic form is universal is given in Conway's book [**3**, p. 42].

In 1938 Halmos [**11**] observed from the work of Ramanujan that a necessary and sufficient condition for the positive diagonal quaternary integral quadratic form $a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2$ to be universal is that it represents the first fifteen positive integers. This condition provides a very convenient way of determining whether a given form $a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2$ is universal or not. Instead of remembering a list of 54 forms, or looking the list up, we have only to check whether the form of interest represents each of the integers $1, 2, \ldots, 15$. For example it is easily checked that the form $x_1^2 + 2x_2^2 + 4x_3^2 + 14x_4^2$ does represent each of these integers and so is universal, whereas the form $x_1^2 + 2x_2^2 + 5x_3^2 + 5x_4^2$ does not represent 15 and so is clearly not universal. In fact the set $\{1, 2, \ldots, 15\}$ is not minimal. It is enough to check the representability of each integer in the set $\{1, 2, 3, 5, 6, 7, 10, 14, 15\}$ in order to determine whether the form is universal or not. This set is minimal in the sense that if $m \in \{1, 2, 3, 5, 6, 7, 10, 14, 15\}$, then there is a diagonal form $a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2$ that does not represent m but represents every other positive integer. Such forms were called almost-universal forms by Halmos [**11**]. He showed for example that the form $2x_1^2 + 3x_2^2 + 4x_3^2 + 5x_4^2$ represents every positive integer except 1.

The observation of Halmos was not pursued for many years until 1993 when it was taken up by Conway in a graduate course on quadratic forms at Princeton University. Conway and his student Schneeberger were able to extend the "15 theorem" from positive diagonal quaternary integral quadratic forms to all positive integer-matrix quadratic forms. They proved but did not publish the proof of the following theorem.

15-Theorem. *If a positive integer-matrix quadratic form in any number of variables represents all the positive integers up to and including* 15 *then it is universal.*

The details of their work are given in [**4, 22**]. In 2000 Bhargava [**1**] gave a new, beautiful, brilliant proof of the 15-theorem in the following stronger form.

Strong 15-Theorem. *If a positive integer-matrix quadratic form in any number of variables represents all the nine integers*

$$1, 2, 3, 5, 6, 7, 10, 14, 15$$

then it is universal.

Bhargava showed that the set $\{1, 2, 3, 5, 6, 7, 10, 14, 15\}$ is minimal in the sense that if m is any one of these numbers then there is a positive diagonal quaternary integral quadratic form that fails to represent m but represents every positive integer different from m. Bhargava also established that there are exactly 204 positive universal quaternary integer-matrix quadratic forms. This corrected the work of Willerding [**23, 24**].

In 1993 Conway formulated the conjecture that a positive integral quadratic form that represents all the positive integers up to and including 290 must be universal. This was proved by Bhargava and Hanke [**2**].

290-Theorem. *If a positive integral quadratic form in any number of variables represents all the positive integers up to and including* 290*, then it is universal.*

Indeed Bhargava and Hanke proved this result in the following stronger form.

Strong 290-Theorem. *If a positive integral quadratic form in any number of variables represents all the twenty-nine integers*

$$1, 2, 3, 5, 6, 7, 10, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29,$$

$$30, 31, 34, 35, 37, 42, 58, 93, 110, 145, 203, 290$$

then it is universal.

Bhargava and Hanke showed too that the set in the strong 290-theorem is minimal in the sense that we have already explained. The 290-theorem allowed Bhargava and Hanke to determine all positive universal quaternary integral quadratic forms. They found that there are exactly 6436 such forms. Dickson's theorems [**7**] on universal quaternary quadratic integral forms follow from the Strong 290-theorem.

A positive integral quadratic form that represents every positive integer in the arithmetic progression $\{km + \ell \mid m = 0, 1, 2, \ldots\}$, where k and ℓ are positive integers with $1 \leqslant \ell \leqslant k$ is called (k, ℓ)-universal. Thus a positive integral quadratic form that represents all odd natural numbers is $(2, 1)$-universal. No positive ternary integral quadratic form can be (k, k)-universal for any positive integer k. To see this, suppose on the contrary that a positive ternary integral quadratic form $q(x_1, x_2, x_3)$ represents all positive multiples of the positive integer k. Then the rational positive ternary quadratic form $\frac{1}{k}q(x_1, x_2, x_3)$ represents all positive integers. But it is known that a rational positive ternary quadratic form fails to represent rationally some full congruence class of integers. This is the required contradiction. This argument can be found for example in Conway's delightful book [**3**, p. 142], see also [**3**, pp. 81–83] for a related amusing theorem concerning the little Methuselah form $x^2 + 2y^2 + yz + 4z^2$. Thus, in particular, a positive ternary integral quadratic form cannot represent every positive even integer. A proof of this for diagonal ternaries was given by Panaitopol [**19**]. Euler conjectured in 1748 that the positive ternary integral quadratic form $x_1^2 + x_2^2 + 2x_3^2$ represents every positive odd integer, see [**10**, p. 206]. Dickson [**6**] proved that the positive diagonal ternary integral quadratic forms

$$x_1^2 + x_2^2 + 2x_3^2, \quad x_1^2 + 2x_2^2 + 3x_3^2, \quad x_1^2 + 2x_2^2 + 4x_3^2 \tag{1}$$

represent all positive odd integers. More generally, in 1995 Kaplansky [**13**] gave a list of 23 positive ternary integral quadratic forms which must contain all the (2,1)-universal ternary integral quadratic forms. He proved the (2, 1)-universality of 19 of the 23 forms in his list. All of the diagonal ternaries in his list were among the 19 for which he proved (2, 1)-universality. These were precisely the three listed in (1). Thus the only positive diagonal ternary quadratic integral forms that represent all positive odd integers are the three listed in (1). Another proof of this has been given by Panaitopol [**19**]. In 1996 Jagy [**12**] proved that one of the Kaplansky's four leftover forms, namely $x^2 + 3y^2 + 11z^2 + xy + 7yz$, is (2, 1)-universal. It appears to be very difficult to decide whether the remaining three forms

$$x^2 + 2y^2 + 5z^2 + xz, \quad x^2 + 3y^2 + 6z^2 + xy + 2yz, \quad x^2 + 3y^2 + 7z^2 + xy + xz \tag{2}$$

are (2, 1)-universal or not. Rouse [**21**] remarks that at present there is no general algorithm for determining the integers represented by a positive ternary integral quadratic form. Assuming that the three forms in (2) do in fact represent all positive odd integers, Rouse [**21**] has shown that a positive integral quadratic form in any number of variables is (2, 1)-universal if and only if it represents the positive odd integers 1 to 451 inclusive.

451-Theorem. *Assuming that the three ternary forms in* (2) *represent all positive odd integers, then a positive integral quadratic form in any number of variables is* (2, 1)*-universal if and only if it represents all the odd integers from* 1 *to* 451 *inclusive*.

Rouse's main result [**21**] was the minimal set of positive odd integers needed for (2, 1)-universality.

Strong 451-Theorem. *Assuming that the three ternary forms in* (2) *represent all positive odd integers, then a positive integral quadratic form in any number of variables is* (2, 1)*-universal if and only if it represents all of the* 46 *integers:*

1, 3, 5, 7, 11, 13, 15, 17, 19, 21, 23, 29, 31, 33, 35, 37, 39, 41, 47,

51, 53, 57, 59, 77, 83, 85, 87, 89, 91, 93, 105, 119, 123, 133, 137,

143, 145, 187, 195, 203, 205, 209, 231, 319, 385, 451.

Prior to Rouse's work, Bhargava had shown (but not published) that a positive integer-matrix quadratic form in any number of variables is (2, 1)-universal if and only if it represents all of the seven integers 1, 3, 5, 7, 11, 15, 33. This 33-theorem is stated by Conway [**4**]. If we restrict attention to positive diagonal ternaries, then we know which ones are (2, 1)-universal from the work of Dickson and Kaplansky and we can determine the subset of the set in the 33-theorem (or the strong 451-theorem) which they must represent in order to be recognized as (2, 1)-universal. Our "four integers" theorem gives this subset.

Theorem A: Four Integers Theorem. *If the ternary quadratic form $ax^2 + by^2 + cz^2$, where a, b, and c are positive integers, represents the four integers*

$$1,\ 3,\ 5 \textit{ and } 15,$$

then it is (2, 1)*-universal.*

We also show that there is a "five integers" theorem for positive diagonal ternary integral quadratic forms representing all positive integers which are congruent to 2 modulo 4.

Theorem B: Five Integers Theorem. *If the ternary quadratic form $ax^2 + by^2 + cz^2$, where a, b, and c are positive integers, represents the five integers*

$$2,\ 6,\ 10,\ 14 \textit{ and } 30,$$

then it is (4, 2)*-universal.*

As we have already mentioned, there are precisely three ternary quadratic forms $ax^2 + by^2 + cz^2$ (a, b, c positive integers with $a \leqslant b \leqslant c$) representing every odd natural number, namely $x^2 + y^2 + 2z^2$, $x^2 + 2y^2 + 3z^2$, and $x^2 + 2y^2 + 4z^2$. We will show that there are exactly nine such forms representing every natural number $\equiv 2 \pmod 4$, namely $x^2 + y^2 + z^2$, $x^2 + y^2 + 4z^2$, $x^2 + y^2 + 5z^2$, $x^2 + 2y^2 + 2z^2$, $x^2 + 2y^2 + 6z^2$, $x^2 + 2y^2 + 8z^2$, $2x^2 + 2y^2 + 4z^2$, $2x^2 + 4y^2 + 6z^2$, and $2x^2 + 4y^2 + 8z^2$.

The following simple lemma enables us to bound the coefficients of a ternary quadratic form $ax^2 + by^2 + cz^2$ in terms of the integers it represents. We state the result more generally for an arbitrary positive diagonal integral quadratic form.

Bounding Lemma. *Let k be a positive integer. Let $a_1, \ldots, a_k$ be positive integers with $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_k$. Set*

$$q := a_1x_1^2 + \cdots + a_kx_k^2, \quad q_1 := 0,$$

and for $i = 2, \ldots, k$ let

$$q_i := a_1x_1^2 + \cdots + a_{i-1}x_{i-1}^2.$$

If q represents a positive integer n, but for some $i \in \{1, 2, \ldots, k\}$ q_i does not represent n, then $a_i \leqslant n$.

Proof. As q represents n there are integers $y_1, \ldots, y_k$ such that

$$n = a_1y_1^2 + \cdots + a_ky_k^2.$$

If $(y_i, \ldots, y_k) = (0, \ldots, 0)$, then

$$n = \begin{cases} a_1 y_1^2 + \cdots + a_{i-1} y_{i-1}^2 & \text{if } 2 \leqslant i \leqslant k, \\ 0 & \text{if } i = 1, \end{cases}$$

contradicting that q_i does not represent n when $i \geqslant 2$ and contradicting that $n > 0$ when $i = 1$. Hence $(y_i, \ldots, y_k) \neq (0, \ldots, 0)$ and so $y_i^2 + \cdots + y_k^2 \geqslant 1$. Thus

$$n = a_1 y_1^2 + \cdots + a_k y_k^2 \geqslant a_i y_i^2 + \cdots + a_k y_k^2 \geqslant a_i (y_i^2 + \cdots + y_k^2) \geqslant a_i,$$

as claimed. ∎

We make extensive use of the bounding lemma in the proofs of Theorems A and B.

Proof of Theorem A. Suppose that the ternary quadratic form $ax^2 + by^2 + cz^2$, where a, b, and c are positive integers, represents the integers 1, 3, 5, and 15. Without loss of generality we may suppose that $a \leqslant b \leqslant c$.

As $ax^2 + by^2 + cz^2$ represents 1, by the bounding lemma we have $a = 1$. As $x^2 + by^2 + cz^2$ represents 3 and $3 \neq r^2$ for any integer r, by the bounding lemma we have $1 \leqslant b \leqslant 3$. When $b = 1$, as $x^2 + y^2 + cz^2$ represents 3 and $3 \neq r^2 + s^2$ for any integers r and s, by the bounding lemma we have $1 \leqslant c \leqslant 3$. When $b = 2$, as $x^2 + 2y^2 + cz^2$ represents 5 and $5 \neq r^2 + 2s^2$ for any integers r and s, by the bounding lemma we have $2 \leqslant c \leqslant 5$. When $b = 3$ as $x^2 + 3y^2 + cz^2$ represents 5 and $5 \neq r^2 + 3s^2$ for any integers r and s, by the bounding lemma we have $3 \leqslant c \leqslant 5$. Thus we have 10 forms to examine, namely

(A)	$x^2 + y^2 + z^2$	(B)	$x^2 + y^2 + 2z^2$	(C)	$x^2 + y^2 + 3z^2$
(D)	$x^2 + 2y^2 + 2z^2$	(E)	$x^2 + 2y^2 + 3z^2$	(F)	$x^2 + 2y^2 + 4z^2$
(G)	$x^2 + 2y^2 + 5z^2$	(H)	$x^2 + 3y^2 + 3z^2$	(I)	$x^2 + 3y^2 + 4z^2$
(J)	$x^2 + 3y^2 + 5z^2$.				

The forms (A), (C), (D), (G), (I), and (J) do not represent 15. The form (H) does not represent 5. This leaves only the forms (B), (E), and (F). By the aforementioned results of Dickson and Kaplansky, these three forms represent all positive odd integers. This completes the proof of Theorem A. ∎

The set $\{1, 3, 5, 15\}$ is minimal as $2x^2 + 3y^2 + 4z^2$ represents 3, 5, and 15 but not 1, $x^2 + y^2 + 5z^2$ represents 1, 5, and 15 but not 3, $x^2 + 2y^2 + 6z^2$ represents 1, 3, and 15 but not 5, and $x^2 + y^2 + z^2$ represents 1, 3, and 5 but not 15.

To prove Theorem B we require the following results. For the proofs of these results we refer the reader to [**6**], [**8**], and [**9**, Chapter 5].

(I) A positive integer n is represented by the form $x^2 + y^2 + z^2$ if and only if $n \neq 4^k(8l + 7)$ for any nonnegative integers k and l.

(II) A positive integer n is represented by the form $x^2 + y^2 + 4z^2$ if and only if $n \neq 8l + 3$ and $n \neq 4^k(8l + 7)$ for any nonnegative integers k and l.

(III) A positive integer n is represented by the form $x^2 + y^2 + 5z^2$ if and only if $n \neq 4^k(8l + 3)$ for any nonnegative integers k and l.

(IV) A positive integer n is represented by the form $x^2 + 2y^2 + 2z^2$ if and only if $n \neq 4^k(8l + 7)$ for any nonnegative integers k and l.

(V) A positive integer n is represented by the form $x^2 + 2y^2 + 6z^2$ if and only if $n \neq 4^k(8l + 5)$ for any nonnegative integers k and l.

(VI) A positive integer n is represented by the form $x^2 + 2y^2 + 8z^2$ if and only if $n \neq 8l + 5$ and $n \neq 4^k(8l + 7)$ for any nonnegative integers k and l.

Proof of Theorem B. Suppose that $ax^2 + by^2 + cz^2$, where a, b, and c are positive integers, represents the integers 2, 6, 10, 14, and 30. Without loss of generality we may suppose that $a \leqslant b \leqslant c$.

As $ax^2 + by^2 + cz^2$ represents 2, by the bounding lemma we have $1 \leqslant a \leqslant 2$.

When $a = 1$, as $x^2 + by^2 + cz^2$ represents 2 and $2 \neq r^2$ for any integer r, by the bounding lemma we have $1 \leqslant b \leqslant 2$. When $a = 1$ and $b = 1$, as $x^2 + y^2 + cz^2$ represents 6 and $6 \neq r^2 + s^2$ for any integers r and s, by the bounding lemma we have $1 \leqslant c \leqslant 6$. When $a = 1$ and $b = 2$ as $x^2 + 2y^2 + cz^2$ represents 10 and $10 \neq r^2 + 2s^2$ for any integers r and s, by the bounding lemma we have $2 \leqslant c \leqslant 10$.

When $a = 2$, as $2x^2 + by^2 + cz^2$ represents 6 and $6 \neq 2r^2$ for any integer r, by the bounding lemma we have $2 \leqslant b \leqslant 6$. When $a = 2$ and $b = 2$, as $2x^2 + 2y^2 + cz^2$ represents 6 and $6 \neq 2r^2 + 2s^2$ for any integers r and s, by the bounding lemma we have $2 \leqslant c \leqslant 6$. When $a = 2$ and $b = 3$, as $2x^2 + 3y^2 + cz^2$ represents 6 and $6 \neq 2r^2 + 3s^2$ for any integers r and s, by the bounding lemma we have $3 \leqslant c \leqslant 6$. When $a = 2$ and $b = 4$, as $2x^2 + 4y^2 + cz^2$ represents 10 and $10 \neq 2r^2 + 4s^2$ for any integers r and s, by the bounding lemma we have $4 \leqslant c \leqslant 10$. When $a = 2$ and $b = 5$, as $2x^2 + 5y^2 + cz^2$ represents 6 and $6 \neq 2r^2 + 5s^2$ for any integers r and s, we have $5 \leqslant c \leqslant 6$. When $a = 2$ and $b = 6$, as $2x^2 + 6y^2 + cz^2$ represents 10 and $10 \neq 2r^2 + 6s^2$ for any integers r and s, we have $6 \leqslant c \leqslant 10$. Hence we have 38 forms to examine:

(1)	$x^2 + y^2 + z^2$	(2)	$x^2 + y^2 + 2z^2$	(3)	$x^2 + y^2 + 3z^2$
(4)	$x^2 + y^2 + 4z^2$	(5)	$x^2 + y^2 + 5z^2$	(6)	$x^2 + y^2 + 6z^2$
(7)	$x^2 + 2y^2 + 2z^2$	(8)	$x^2 + 2y^2 + 3z^2$	(9)	$x^2 + 2y^2 + 4z^2$
(10)	$x^2 + 2y^2 + 5z^2$	(11)	$x^2 + 2y^2 + 6z^2$	(12)	$x^2 + 2y^2 + 7z^2$
(13)	$x^2 + 2y^2 + 8z^2$	(14)	$x^2 + 2y^2 + 9z^2$	(15)	$x^2 + 2y^2 + 10z^2$
(16)	$2x^2 + 2y^2 + 2z^2$	(17)	$2x^2 + 2y^2 + 3z^2$	(18)	$2x^2 + 2y^2 + 4z^2$
(19)	$2x^2 + 2y^2 + 5z^2$	(20)	$2x^2 + 2y^2 + 6z^2$	(21)	$2x^2 + 3y^2 + 3z^2$
(22)	$2x^2 + 3y^2 + 4z^2$	(23)	$2x^2 + 3y^2 + 5z^2$	(24)	$2x^2 + 3y^2 + 6z^2$
(25)	$2x^2 + 4y^2 + 4z^2$	(26)	$2x^2 + 4y^2 + 5z^2$	(27)	$2x^2 + 4y^2 + 6z^2$
(28)	$2x^2 + 4y^2 + 7z^2$	(29)	$2x^2 + 4y^2 + 8z^2$	(30)	$2x^2 + 4y^2 + 9z^2$
(31)	$2x^2 + 4y^2 + 10z^2$	(32)	$2x^2 + 5y^2 + 5z^2$	(33)	$2x^2 + 5y^2 + 6z^2$
(34)	$2x^2 + 6y^2 + 6z^2$	(35)	$2x^2 + 6y^2 + 7z^2$	(36)	$2x^2 + 6y^2 + 8z^2$
(37)	$2x^2 + 6y^2 + 9z^2$	(38)	$2x^2 + 6y^2 + 10z^2$.		

Forms (3), (17), (19), (23), and (32) do not represent 6. Forms (8), (10), (21), (22), (24), (26), (28), (30), (33), (34), (35), and (37) do not represent 10. Forms (2), (9), (12), (14), (16), and (25) do not represent 14. Forms (6), (15), (20), (31), (36), and (38) do not represent 30.

This leaves the 9 forms (1), (4), (5), (7), (11), (13), (18), (27), and (29). By Theorem 1 the forms $x^2 + y^2 + 2z^2$, $x^2 + 2y^2 + 3z^2$, and $x^2 + 2y^2 + 4z^2$ represent every odd positive integer. Hence their doubles, namely forms (18), (27), and (29), represent every positive integer congruent to 2 modulo 4. Let $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Let $m \in \mathbb{N}_0$. As $4m + 2 \neq 4^k(8l + 7)$ for any $k, l \in \mathbb{N}_0$, by (I) the form (1) represents every positive integer congruent to 2 modulo 4. As $4m + 2 \neq 8l + 3, 4^k(8l + 7)$ for any $k, l \in \mathbb{N}_0$, by (II) the form (4) represents every positive integer congruent to 2 modulo 4. As $4m + 2 \neq 4^k(8l + 3)$ for any $k, l \in \mathbb{N}_0$, by (III) the form (5) represents every positive integer congruent to 2 modulo 4. As $4m + 2 \neq 4^k(8l + 7)$ for any $k, l \in \mathbb{N}_0$, by (IV) the form (7) represents every positive integer congruent to 2 modulo 4. As $4m + 2 \neq$

$4^k(8l+5)$ for any $k, l \in \mathbb{N}_0$, by (V) the form (11) represents every positive integer congruent to 2 modulo 4. As $4m+2 \neq 8l+5, 4^k(8l+7)$ for any $k, l \in \mathbb{N}_0$, by (VI) the form (13) represents every positive integer congruent to 2 modulo 4.

This completes the proof of Theorem B. ■

The set $\{2, 6, 10, 14, 30\}$ is minimal as $x^2+5y^2+5z^2$ represents 6, 10, 14, and 30 but not 2, $x^2+y^2+3z^2$ represents 2, 10, 14, and 30 but not 6, $x^2+2y^2+3z^2$ represents 2, 6, 14, and 30 but not 10, $x^2+2y^2+7z^2$ represents 2, 6, 10, and 30 but not 14, and $x^2+y^2+6z^2$ represents 2, 6, 10, and 14 but not 30.

It is natural to ask whether the elementary approach used in this article to determine all positive diagonal ternary integral quadratic forms which are (2,1)-universal and (4,2)-universal can also be used to find those such ternaries that are (8,4)-universal. In attempting to do this, one encounters the form $x^2+y^2+11z^2$, which represents every positive integer $\equiv 4 \pmod 8$ up to 300 but not 308, as well as the form $x^2+2y^2+9z^2$, which appears to represent every positive integer $\equiv 4 \pmod 8$. The proof of the (8, 4)-universality of the form $x^2+2y^2+9z^2$ may be difficult to prove. Assuming that the first form in (2) represents all odd positive integers, the author (Acta Arithmetica 166.4 (2014), 391–396) has determined all positive diagonal ternary quadratic forms which are (8,4)-universal. It turned out that proving the (8,4)-universality of the form just mentioned was not difficult.

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100 Years Ago This Month in The American Mathematical Monthly
Edited by Vadim Ponomarenko

Professor H. S. WHITE, of Vassar College, was elected a member of the National Academy of Sciences at the annual meeting held last April in Washington, D.C. The other members representing pure mathematics in this Academy are Professors BÔCHER, BOLZA, DICKSON, MOORE, OSGOOD, STOREY, and VAN VLECK.

The cover for the May issue was dated "April" by an oversight of the printer. The printed slip herewith may be used to correct the error. The next issue of the MONTHLY, will be for September, 1915.

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