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SEXTENARY QUADRATIC FORMS AND AN IDENTITY OF KLEIN AND FRICKE

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Formulae, originally conjectured by Liouville, are proved for the number of representations of a positive integer n by each of the eight sextenary quadratic forms $x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+4x_6^2$, $x_1^2+x_2^2+x_3^2+x_4^2+4x_5^2+4x_6^2$, $x_1^2+x_2^2+x_3^2+4x_4^2+4x_5^2+4x_6^2$, $x_1^2+x_2^2+4x_3^2+4x_4^2+4x_5^2+4x_6^2$, $x_1^2+4x_2^2+4x_3^2+4x_4^2+4x_5^2+4x_6^2$, $x_1^2+x_2^2+x_3^2+2x_4^2+2x_5^2+4x_6^2$, $x_1^2+x_2^2+2x_3^2+2x_4^2+4x_5^2+4x_6^2$, $x_1^2+2x_2^2+2x_3^2+4x_4^2+4x_5^2+4x_6^2$.

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1. Introduction

Let $\mathbb{Z}$ and $\mathbb{N}$ denote the sets of integers and positive integers, respectively. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $n \in \mathbb{N}_0$ and $a_1, \ldots, a_6 \in \mathbb{N}$ we let

$$N(a_1, \ldots, a_6; n) := \text{card}\{(x_1, \ldots, x_6) \in \mathbb{Z}^6 \mid n = a_1x_1^2 + \cdots + a_6x_6^2\}. \tag{1.1}$$

Clearly

$$N(a_1, \ldots, a_6; 0) = 1. \tag{1.2}$$

For $n \in \mathbb{N}$, we define

$$G_4(n) := \sum_{d \mid n} \left(\frac{-4}{n/d}\right) d^2, \quad H_4(n) := \sum_{d \mid n} \left(\frac{-4}{d}\right) d^2, \tag{1.3}$$

where d runs through the positive integers dividing n and $\left(\frac{-4}{k}\right)(k \in \mathbb{N})$ is the Legendre–Jacobi–Kronecker symbol for discriminant -4, that is

$$\left(\frac{-4}{k}\right) = \begin{cases} 1, & \text{if } k \equiv 1 \pmod 4, \\ -1, & \text{if } k \equiv 3 \pmod 4, \\ 0, & \text{if } k \equiv 0 \pmod 2. \end{cases} \tag{1.4}$$

It is easy to show that

$$G_4(2n) = 4G_4(n), \quad H_4(2n) = H_4(n), \quad n \in \mathbb{N}. \tag{1.5}$$

In [3], we proved the following formulae:

$$N(1,1,1,1,1,1;n) = 16G_4(n) - 4H_4(n), \tag{1.6}$$

$$N(1,1,1,1,2,2;n) = 8G_4(n) - 2(1+(-1)^n)H_4(n), \tag{1.7}$$

$$N(1,1,2,2,2,2;n) = 4G_4(n) - 2(1+(-1)^n)H_4(n), \tag{1.8}$$

and

$$N(1,2,2,2,2,4;n) = \begin{cases} 2G_4(n), & \text{if } n \not\equiv 0 \pmod 4, \\ 2G_4(n) - 4H_4(n/4), & \text{if } n \equiv 0 \pmod 4. \end{cases} \tag{1.9}$$

The formula (1.6) is the classical formula of Jacobi [5, §§40–42] for the number of representations of n ($\in \mathbb{N}$) as a sum of six squares. The remaining three formulae were stated but not proved by Liouville [7–9]. In this paper, we obtain similar formulae for

$$N(1,1,1,1,1,4;n),\ N(1,1,1,1,4,4;n),\ N(1,1,1,4,4,4;n),\ N(1,1,4,4,4,4;n),$$
$$N(1,4,4,4,4,4;n),\ N(1,1,1,2,2,4;n),\ N(1,1,2,2,4,4;n),\ N(1,2,2,4,4,4;n),$$

valid for all $n \in \mathbb{N}$. These formulae are given in terms of $G_4(n)$ and $H_4(n)$ when $n \not\equiv 1 \pmod 4$. However, when $n \equiv 1 \pmod 4$, in addition to $G_4(n)$ and $H_4(n)$, we require the function

$$I(n) := \sum_{\substack{(x,y)\in\mathbb{Z}^2 \\ n=x^2+4y^2}} (x^2 - 4y^2), \quad n \equiv 1 \pmod 2. \tag{1.10}$$

We note that $x^2 + 4y^2 \equiv 0$ or $1 \pmod 4$ for $(x, y) \in \mathbb{Z}^2$ so that

$$I(n) = 0, \quad \text{if } n \equiv 3 \pmod 4. \tag{1.11}$$

Let $\mathbb{C}$ denote the field of complex numbers. The basic property of $I(n)$ that we shall use to prove the conjectured formulae of Liouville [10–17] is

$$\sum_{\substack{n=1 \\ n\equiv 1 \pmod 4}}^{\infty} I(n)q^n = 2q \prod_{n=1}^{\infty} (1 - q^{4n})^6, \quad q \in \mathbb{C}, \quad |q| < 1, \tag{1.12}$$

which can be found in [6, Vol. 2, p. 377] and [18, p. 122]. We prove

Theorem 1.1. *Let $n \in \mathbb{N}$. Then*

$$\text{(i)}\quad N(1,1,1,1,1,4;n) = \begin{cases} 6G_4(n) + 2I(n), & \text{if } n \equiv 1 \pmod 4, \\ 10G_4(n), & \text{if } n \equiv 2,3 \pmod 4, \\ 6G_4(n) - 4H_4(n), & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

$$\text{(ii)}\quad N(1,1,1,1,4,4;n) = \begin{cases} 4G_4(n) + 2I(n), & \text{if } n \equiv 1 \pmod 4, \\ 4G_4(n), & \text{if } n \equiv 3 \pmod 4, \\ 6G_4(n), & \text{if } n \equiv 2 \pmod 4, \\ 2G_4(n) - 4H_4(n), & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

$$\text{(iii)}\quad N(1,1,1,4,4,4;n) = \begin{cases} 3G_4(n) + \frac{3}{2}I(n), & \text{if } n \equiv 1 \pmod 4, \\ G_4(n), & \text{if } n \equiv 3 \pmod 4, \\ 3G_4(n), & \text{if } n \equiv 2 \pmod 4, \\ G_4(n) - 4H_4(n), & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

$$\text{(iv)}\quad N(1,1,4,4,4,4;n) = \begin{cases} 2G_4(n) + I(n), & \text{if } n \equiv 1 \pmod 4, \\ 0, & \text{if } n \equiv 3 \pmod 4, \\ G_4(n), & \text{if } n \equiv 2 \pmod 4, \\ G_4(n) - 4H_4(n), & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

$$\text{(v)}\quad N(1,4,4,4,4,4;n) = \begin{cases} G_4(n) + \frac{1}{2}I(n), & \text{if } n \equiv 1 \pmod 4, \\ 0, & \text{if } n \equiv 2,3 \pmod 4, \\ G_4(n) - 4H_4(n), & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

$$\text{(vi)}\quad N(1,1,1,2,2,4;n) = \begin{cases} 4G_4(n) + I(n), & \text{if } n \equiv 1 \pmod 4, \\ 4G_4(n), & \text{if } n \equiv 2,3 \pmod 4, \\ 4G_4(n) - 4H_4(n), & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

$$\text{(vii)}\quad N(1,1,2,2,4,4;n) = \begin{cases} 2G_4(n) + I(n), & \text{if } n \equiv 1 \pmod 4, \\ 2G_4(n), & \text{if } n \equiv 2,3 \pmod 4, \\ 2G_4(n) - 4H_4(n), & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

$$\text{(viii)}\quad N(1,2,2,4,4,4;n) = \begin{cases} G_4(n) + \frac{1}{2}I(n), & \text{if } n \equiv 1 \pmod 4, \\ G_4(n), & \text{if } n \equiv 2,3 \pmod 4, \\ G_4(n) - 4H_4(n), & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

2. Preliminary Results

For $q \in \mathbb{C}$ with $|q| < 1$ we set

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} \tag{2.1}$$

and

$$\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2}. \tag{2.2}$$

The basic properties of $\varphi(q)$ are (see, for example, [4, pp. 15, 71, 72])

$$\varphi(q) + \varphi(-q) = 2\varphi(q^4), \tag{2.3}$$

$$\varphi^2(q) + \varphi^2(-q) = 2\varphi^2(q^2), \tag{2.4}$$

$$\varphi(q)\varphi(-q) = \varphi^2(-q^2), \tag{2.5}$$

$$\varphi(q) - \varphi(-q) = 4q\psi(q^8). \tag{2.6}$$

Jacobi [5] (see, for example, [4, Corollary 1.3.4 and Theorem 1.3.10]) has shown that

$$\varphi(q) = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^5}{(1-q^n)^2(1-q^{4n})^2}, \tag{2.7}$$

$$\varphi(-q) = \prod_{n=1}^{\infty} \frac{(1-q^n)^2}{(1-q^{2n})}, \tag{2.8}$$

and

$$\psi(q) = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^2}{(1-q^n)}. \tag{2.9}$$

Theorem 2.1. *For $q \in \mathbb{C}$ with $|q| < 1$, we have*

$$\sum_{\substack{n=1 \\ n\equiv 1 \pmod 4}}^{\infty} I(n)q^n = \frac{1}{8}\varphi^5(q)\varphi(-q) - \frac{1}{8}\varphi(q)\varphi^5(-q).$$

Proof. From (2.7) and (2.9), we see that

$$\varphi(q^2) = \prod_{n=1}^{\infty} \frac{(1-q^{4n})^5}{(1-q^{2n})^2(1-q^{8n})^2},$$

$$\varphi(q^4) = \prod_{n=1}^{\infty} \frac{(1-q^{8n})^5}{(1-q^{4n})^2(1-q^{16n})^2},$$

and

$$\psi(q^8) = \prod_{n=1}^{\infty} \frac{(1-q^{16n})^2}{(1-q^{8n})},$$

so that

$$\varphi(q)\varphi(-q)\varphi^2(q^2)\varphi(q^4)\psi(q^8) = \prod_{n=1}^{\infty} (1-q^{4n})^6.$$

Hence, by (1.12), we have

$$\begin{aligned}\sum_{\substack{n=1\\ n\equiv 1 \pmod 4}}^{\infty} I(n)q^n &= 2q\varphi(q)\varphi(-q)\varphi^2(q^2)\varphi(q^4)\psi(q^8)\\ &= \frac{1}{8}\varphi(q)\varphi(-q)(\varphi^2(q)+\varphi^2(-q))(\varphi(q)+\varphi(-q))(\varphi(q)-\varphi(-q))\\ &= \frac{1}{8}\varphi(q)\varphi(-q)(\varphi^4(q)-\varphi^4(-q))\\ &= \frac{1}{8}\varphi^5(q)\varphi(-q) - \frac{1}{8}\varphi(q)\varphi^5(-q),\end{aligned}$$

as asserted. □

Theorem 2.2. *For $q \in \mathbb{C}$ with $|q| < 1$ we have*

(i) $\displaystyle\sum_{n=1}^{\infty} G_4(n)q^n = \frac{1}{16}\varphi^6(q) - \frac{1}{16}\varphi^2(q)\varphi^4(-q),$

(ii) $\displaystyle 1 - 4\sum_{n=1}^{\infty} H_4(n)q^n = \varphi^2(q)\varphi^4(-q).$

Proof. Part (i) is [2, Lemma 2] and part (ii) is [2, Lemma 1]. □

Theorem 2.3. *For $q \in \mathbb{C}$ with $|q| < 1$, we have*

$$\sum_{\substack{n=1\\ n\equiv 1 \pmod 2}}^{\infty} G_4(n)q^n = q\varphi^4(q^2)\varphi(q^4)\psi(q^8).$$

Proof. Appealing to Theorems 2.2(i), (2.3), (2.4) and (2.6), we obtain

$$\begin{aligned}\sum_{\substack{n=1\\ n\equiv 1 \pmod 2}}^{\infty} G_4(n)q^n &= \frac{1}{2}\sum_{n=1}^{\infty} G_4(n)q^n - \frac{1}{2}\sum_{n=1}^{\infty} G_4(n)(-q)^n\\ &= \frac{1}{32}\varphi^6(q) - \frac{1}{32}\varphi^2(q)\varphi^4(-q) - \frac{1}{32}\varphi^6(-q)\\ &\quad + \frac{1}{32}\varphi^4(q)\varphi^2(-q)\end{aligned}$$

$$= \frac{1}{32}(\varphi(q)-\varphi(-q))(\varphi(q)+\varphi(-q))(\varphi^2(q)+\varphi^2(-q))^2$$
$$= q\varphi^4(q^2)\varphi(q^4)\psi(q^8). \qquad \square$$

Theorem 2.4. *For $q \in \mathbb{C}$ with $|q| < 1$, we have*

$$\sum_{\substack{n=1 \\ n \equiv 2 \pmod 4}}^{\infty} G_4(n)q^n = 4q^2\varphi^4(q^4)\varphi(q^8)\psi(q^{16}).$$

Proof. Appealing to (1.5) and Theorem 2.3, we deduce

$$\sum_{\substack{n=1 \\ n \equiv 2 \pmod 4}}^{\infty} G_4(n)q^n = \sum_{\substack{n=1 \\ n \equiv 1 \pmod 2}}^{\infty} G_4(2n)q^{2n}$$
$$= 4 \sum_{\substack{n=1 \\ n \equiv 1 \pmod 2}}^{\infty} G_4(n)q^{2n}$$
$$= 4q^2\varphi^4(q^4)\varphi(q^8)\psi(q^{16}). \qquad \square$$

Theorem 2.5. *For $q \in \mathbb{C}$ with $|q| < 1$, we have*

$$\sum_{\substack{n=1 \\ n \equiv 0 \pmod 4}}^{\infty} G_4(n)q^n = \varphi^6(q^4) - \varphi^2(q^4)\varphi^4(-q^4).$$

Proof. Appealing to (1.5) and Theorem 2.2(i), we obtain

$$\sum_{\substack{n=1 \\ n \equiv 0 \pmod 4}}^{\infty} G_4(n)q^n = \sum_{n=1}^{\infty} G_4(4n)q^{4n}$$
$$= 16 \sum_{n=1}^{\infty} G_4(n)q^{4n}$$
$$= \varphi^6(q^4) - \varphi^2(q^4)\varphi^4(-q^4). \qquad \square$$

Theorem 2.6. *For $q \in \mathbb{C}$ with $|q| < 1$, we have*

(i) $$\sum_{\substack{n=1 \\ n \equiv 1 \pmod 4}}^{\infty} G_4(n)q^n = \frac{1}{2}q(\varphi^4(q^2)+\varphi^4(-q^2))\varphi(q^4)\psi(q^8),$$

(ii) $$\sum_{\substack{n=1 \\ n \equiv 3 \pmod 4}}^{\infty} G_4(n)q^n = \frac{1}{2}q(\varphi^4(q^2)-\varphi^4(-q^2))\varphi(q^4)\psi(q^8).$$

Proof. By Theorem 2.3, we have

$$\sum_{\substack{n=1\\ n\equiv 1 \pmod 4}}^{\infty} G_4(n)q^n + \sum_{\substack{n=1\\ n\equiv 3 \pmod 4}}^{\infty} G_4(n)q^n = q\varphi^4(q^2)\varphi(q^4)\psi(q^8). \tag{2.10}$$

Next, by Theorem 2.2(i), we have

$$\begin{aligned}
&\sum_{\substack{n=1\\ n\equiv 1 \pmod 4}}^{\infty} G_4(n)q^n - \sum_{\substack{n=1\\ n\equiv 3 \pmod 4}}^{\infty} G_4(n)q^n \\
&= \frac{1}{2i}\sum_{n=1}^{\infty} G_4(n)((iq)^n - (-iq)^n) \\
&= \frac{1}{32i}(\varphi^6(iq) - \varphi^2(iq)\varphi^4(-iq) - \varphi^6(-iq) + \varphi^2(-iq)\varphi^4(iq)) \\
&= \frac{1}{32i}(\varphi^2(iq) - \varphi^2(-iq))(\varphi^2(iq) + \varphi^2(-iq))^2.
\end{aligned}$$

From [1, Eq. (2.11), p. 145] we have

$$\varphi(iq) = \varphi(q^4) + \frac{i}{2}(\varphi(q) - \varphi(-q))$$

and

$$\varphi(-iq) = \varphi(q^4) - \frac{i}{2}(\varphi(q) - \varphi(-q)),$$

so that by (2.6) we obtain

$$\varphi^2(iq) - \varphi^2(-iq) = 2i(\varphi(q) - \varphi(-q))\varphi(q^4) = 8iq\varphi(q^4)\psi(q^8).$$

By (2.4) (with q replaced by iq), we have

$$\varphi^2(iq) + \varphi^2(-iq) = 2\varphi^2(-q^2).$$

Hence

$$\sum_{\substack{n=1\\ n\equiv 1 \pmod 4}}^{\infty} G_4(n)q^n - \sum_{\substack{n=1\\ n\equiv 3 \pmod 4}}^{\infty} G_4(n)q^n = q\varphi^4(-q^2)\varphi(q^4)\psi(q^8). \tag{2.11}$$

Adding and subtracting (2.10) and (2.11), we obtain (i) and (ii). □

Theorem 2.7. *For $q \in \mathbb{C}$ with $|q| < 1$, we have*

$$1 - 4\sum_{\substack{n=1\\ n\equiv 0 \pmod 4}}^{\infty} H_4(n)q^n = \varphi^2(q^4)\varphi^4(-q^4).$$

Proof. By (1.5) and Theorem 2.2(ii), we have

$$\begin{aligned} 1 - 4 \sum_{\substack{n=1 \\ n \equiv 0 \pmod 4}}^{\infty} H_4(n) q^n &= 1 - 4 \sum_{n=1}^{\infty} H_4(4n) q^{4n} \\ &= 1 - 4 \sum_{n=1}^{\infty} H_4(n) q^{4n} \\ &= \varphi^2(q^4)\varphi^4(-q^4). \end{aligned}$$

□

3. Proof of Theorem 1.1

For convenience, we set

$$\varphi(q) = a, \quad \varphi(q^4) = c. \tag{3.1}$$

By (2.3), we have

$$\varphi(-q) = 2c - a. \tag{3.2}$$

Then by (2.4),

$$\varphi^2(q^2) = \frac{1}{2}(a^2 + (2c-a)^2) = a^2 - 2ac + 2c^2. \tag{3.3}$$

Next, by (2.5),

$$\varphi^2(-q^2) = a(2c - a) = 2ac - a^2. \tag{3.4}$$

Again, by (2.5),

$$\varphi^4(-q^4) = (a^2 - 2ac + 2c^2)(2ac - a^2). \tag{3.5}$$

From (2.6),

$$q\psi(q^8) = \frac{1}{4}a - \frac{1}{4}(2c - a) = \frac{1}{2}a - \frac{1}{2}c. \tag{3.6}$$

From (2.3), (2.6), (3.3) and (3.4), we obtain

$$q^2\varphi(q^8)\psi(q^{16}) = \frac{1}{4}(a^2 - 2ac + c^2). \tag{3.7}$$

Proof of (i). By Theorems 2.7, 2.2(i), 2.5, 2.6(i), 2.1 and Eqs. (3.1)–(3.6), we have

$$\begin{aligned} 1 - 4 \sum_{\substack{n=1 \\ n \equiv 0 \pmod 4}}^{\infty} H_4(n) q^n &+ 10 \sum_{n=1}^{\infty} G_4(n) q^n \\ &- 4 \sum_{\substack{n=1 \\ n \equiv 0 \pmod 4}}^{\infty} G_4(n) q^n - 4 \sum_{\substack{n=1 \\ n \equiv 1 \pmod 4}}^{\infty} G_4(n) q^n + 2 \sum_{\substack{n=1 \\ n \equiv 1 \pmod 4}}^{\infty} I(n) q^n \end{aligned}$$

$$
\begin{aligned}
&= \varphi^2(q^4)\varphi^4(-q^4) + \frac{5}{8}\varphi^6(q) - \frac{5}{8}\varphi^2(q)\varphi^4(-q) \\
&\quad - 4\varphi^6(q^4) + 4\varphi^2(q^4)\varphi^4(-q^4) \\
&\quad - 2q(\varphi^4(q^2) + \varphi^4(-q^2))\varphi(q^4)\psi(q^8) \\
&\quad + \frac{1}{4}\varphi^5(q)\varphi(-q) - \frac{1}{4}\varphi(q)\varphi^5(-q) \\
&= c^2(a^2 - 2ac + 2c^2)(2ac - a^2) + \frac{5}{8}a^6 - \frac{5}{8}a^2(2c-a)^4 \\
&\quad - 4c^6 + 4c^2(a^2 - 2ac + 2c^2)(2ac - a^2) \\
&\quad - (a-c)c(a^2 - 2ac + 2c^2)^2 - (a-c)c(2ac - a^2)^2 \\
&\quad + \frac{1}{4}a^5(2c-a) - \frac{1}{4}a(2c-a)^5 \\
&= a^5c \\
&= \varphi^5(q)\varphi(q^4) \\
&= \sum_{n=0}^{\infty} N(1,1,1,1,1,4;n)q^n.
\end{aligned}
$$

Equating coefficients of q^n $(n \in \mathbb{N})$, we obtain the asserted formula for $N(1,1,1,1,1,4;n)$. □

Proof of (ii). By Theorems 2.7, 2.2(i), 2.5, 2.3, 2.1 and Eqs. (3.1)–(3.6), we obtain

$$
\begin{aligned}
1 - 4 &\sum_{\substack{n=1 \\ n\equiv 0 \pmod 4}}^{\infty} H_4(n)q^n + 6\sum_{n=1}^{\infty} G_4(n)q^n - 4\sum_{\substack{n=1 \\ n\equiv 0 \pmod 4}}^{\infty} G_4(n)q^n \\
&- 2\sum_{\substack{n=1 \\ n\equiv 1 \pmod 2}}^{\infty} G_4(n)q^n + 2\sum_{\substack{n=1 \\ n\equiv 1 \pmod 4}}^{\infty} I(n)q^n \\
&= \varphi^2(q^4)\varphi^4(-q^4) + \frac{3}{8}\varphi^6(q) - \frac{3}{8}\varphi^2(q)\varphi^4(-q) \\
&\quad - 4\varphi^6(q^4) + 4\varphi^2(q^4)\varphi^4(-q^4) - 2q\varphi^4(q^2)\varphi(q^4)\psi(q^8) \\
&\quad + \frac{1}{4}\varphi^5(q)\varphi(-q) - \frac{1}{4}\varphi(q)\varphi^5(-q) \\
&= c^2(a^2 - 2ac + 2c^2)(2ac - a^2) + \frac{3}{8}a^6 - \frac{3}{8}a^2(2c-a)^4 \\
&\quad - 4c^6 + 4c^2(a^2 - 2ac + 2c^2)(2ac - a^2) \\
&\quad - (a^2 - 2ac + c^2)^2c(a-c) + \frac{1}{4}a^5(2c-a) - \frac{1}{4}a(2c-a)^5
\end{aligned}
$$

$$
\begin{aligned}
&= a^4c^2 \\
&= \varphi^4(q)\varphi^2(q^4) \\
&= \sum_{n=0}^{\infty} N(1,1,1,1,4,4;n)q^n.
\end{aligned}
$$

Equating coefficients of q^n ($n \in \mathbb{N}$), we obtain the asserted formula for $N(1,1,1,1,4,4;n)$. □

Proof of (iii). By Theorems 2.7, 2.2(i), 2.6(ii), 2.5, 2.1 and Eqs. (3.1)–(3.6), we deduce

$$
\begin{aligned}
1 - 4 &\sum_{\substack{n=1 \\ n\equiv 0 \pmod 4}}^{\infty} H_4(n)q^n + 3\sum_{n=1}^{\infty} G_4(n)q^n - 2 \sum_{\substack{n=1 \\ n\equiv 3 \pmod 4}}^{\infty} G_4(n)q^n \\
&- 2 \sum_{\substack{n=1 \\ n\equiv 0 \pmod 4}}^{\infty} G_4(n)q^n + \frac{3}{2} \sum_{\substack{n=1 \\ n\equiv 1 \pmod 4}}^{\infty} I(n)q^n \\
= {} & \varphi^2(q^4)\varphi^4(-q^4) + \frac{3}{16}\varphi^6(q) - \frac{3}{16}\varphi^2(q)\varphi^4(-q) \\
&- q(\varphi^4(q^2) - \varphi^4(-q^2))\varphi(q^4)\psi(q^8) - 2\varphi^6(q^4) + 2\varphi^2(q^4)\varphi^4(-q^4) \\
&+ \frac{3}{16}\varphi^5(q)\varphi(-q) - \frac{3}{16}\varphi(q)\varphi^5(-q) \\
= {} & c^2(a^2 - 2ac + 2c^2)(2ac - a^2) + \frac{3}{16}a^6 - \frac{3}{16}a^2(2c-a)^4 \\
&- \frac{1}{2}((a^2 - 2ac + 2c^2)^2 - (2ac - a^2)^2)c(a-c) \\
&- 2c^6 + 2c^2(a^2 - 2ac + 2c^2)(2ac - a^2) \\
&+ \frac{3}{16}a^5(2c-a) - \frac{3}{16}a(2c-a)^5 \\
= {} & a^3c^3 \\
= {} & \varphi^3(q)\varphi^3(q^4) \\
= {} & \sum_{n=0}^{\infty} N(1,1,1,4,4,4;n)q^n.
\end{aligned}
$$

Equating coefficients of q^n ($n \in \mathbb{N}$), we obtain the asserted formula for $N(1,1,1,4,4,4;n)$. □

Proof of (iv). By Theorems 2.7, 2.2(i), 2.6, 2.1 and Eqs. (3.1)–(3.6), we have

$$
\begin{aligned}
1 - 4 &\sum_{\substack{n=1 \\ n\equiv 0 \pmod 4}}^{\infty} H_4(n)q^n + \sum_{n=1}^{\infty} G_4(n)q^n + \sum_{\substack{n=1 \\ n\equiv 1 \pmod 4}}^{\infty} G_4(n)q^n \\
&+ \sum_{\substack{n=1 \\ n\equiv 1 \pmod 4}}^{\infty} I(n)q^n - \sum_{\substack{n=1 \\ n\equiv 3 \pmod 4}}^{\infty} G_4(n)q^n \\
&= \varphi^2(q^4)\varphi^4(-q^4) + \frac{1}{16}\varphi^6(q) - \frac{1}{16}\varphi^2(q)\varphi^4(-q) \\
&\quad + \frac{1}{2}q(\varphi^4(q^2) + \varphi^4(-q^2))\varphi(q^4)\psi(q^8) + \frac{1}{8}\varphi^5(q)\varphi(-q) - \frac{1}{8}\varphi(q)\varphi^5(-q) \\
&\quad - \frac{1}{2}q(\varphi^4(q^2) - \varphi^4(-q^2))\varphi(q^4)\psi(q^8) \\
&= c^2(a^2 - 2ac + 2c^2)(2ac - a^2) + \frac{1}{16}a^6 - \frac{1}{16}a^2(2c-a)^4 \\
&\quad + \frac{1}{4}(a-c)((a^2 - 2ac + 2c^2)^2 + (2ac - a^2)^2)c \\
&\quad + \frac{1}{8}a^5(2c-a) - \frac{1}{8}a(2c-a)^5 - \frac{1}{4}(a-c)((a^2 - 2ac + 2c^2)^2 - (2ac - a^2)^2)c \\
&= a^2c^4 \\
&= \varphi^2(q)\varphi^4(q^4) \\
&= \sum_{n=0}^{\infty} N(1,1,4,4,4,4;n)q^n.
\end{aligned}
$$

Equating coefficients of q^n ($n \in \mathbb{N}$), we obtain the asserted formula for $N(1,1,4,4,4,4;n)$. □

Proof of (v). By Theorems 2.7, 2.2(i), 2.4, 2.6(ii), 2.1 and Eqs. (3.1)–(3.7), we have

$$
\begin{aligned}
1 - 4 &\sum_{\substack{n=1 \\ n\equiv 0 \pmod 4}}^{\infty} H_4(n)q^n + \sum_{n=1}^{\infty} G_4(n)q^n - \sum_{\substack{n=1 \\ n\equiv 2 \pmod 4}}^{\infty} G_4(n)q^n \\
&- \sum_{\substack{n=1 \\ n\equiv 3 \pmod 4}}^{\infty} G_4(n)q^n + \frac{1}{2} \sum_{\substack{n=1 \\ n\equiv 1 \pmod 4}}^{\infty} I(n)q^n
\end{aligned}
$$

$$
\begin{aligned}
&= \varphi^2(q^4)\varphi^4(-q^4) + \frac{1}{16}\varphi^6(q) - \frac{1}{16}\varphi^2(q)\varphi^4(-q)\\
&\quad - 4q^2\varphi^4(q^4)\varphi(q^8)\psi(q^{16}) - \frac{1}{2}q(\varphi^4(q^2) - \varphi^4(-q^2))\varphi(q^4)\psi(q^8)\\
&\quad + \frac{1}{16}\varphi^5(q)\varphi(-q) - \frac{1}{16}\varphi(q)\varphi^5(-q)\\
&= c^2(a^2 - 2ac + 2c^2)(2ac - a^2) + \frac{1}{16}a^6 - \frac{1}{16}a^2(2c - a)^4\\
&\quad - (a^2 - 2ac + c^2)c^4 - \frac{1}{4}(a - c)((a^2 - 2ac + 2c^2)^2 - (2ac - a^2)^2)c\\
&\quad + \frac{1}{16}a^5(2c - a) - \frac{1}{16}a(2c - a)^5\\
&= ac^5\\
&= \varphi(q)\varphi^5(q^4)\\
&= \sum_{n=0}^{\infty} N(1, 4, 4, 4, 4, 4; n)q^n.
\end{aligned}
$$

Equating coefficients of q^n ($n \in \mathbb{N}$), we obtain the asserted formula for $N(1, 4, 4, 4, 4, 4; n)$. □

Proof of (vi). By Theorems 2.7, 2.2(i), 2.1 and Eqs. (3.1)–(3.3), (3.5), we have

$$
\begin{aligned}
&1 - 4\sum_{\substack{n=1\\ n\equiv 0 \pmod 4}}^{\infty} H_4(n)q^n + 4\sum_{n=1}^{\infty} G_4(n)q^n + \sum_{\substack{n=1\\ n\equiv 1 \pmod 4}}^{\infty} I(n)q^n\\
&= \varphi^2(q^4)\varphi^4(-q^4) + \frac{1}{4}\varphi^6(q) - \frac{1}{4}\varphi^2(q)\varphi^4(-q)\\
&\quad + \frac{1}{8}\varphi^5(q)\varphi(-q) - \frac{1}{8}\varphi(q)\varphi^5(-q)\\
&= c^2(a^2 - 2ac + 2c^2)(2ac - a^2) + \frac{1}{4}a^6 - \frac{1}{4}a^2(2c - a)^4\\
&\quad + \frac{1}{8}a^5(2c - a) - \frac{1}{8}a(2c - a)^5\\
&= a^3(a^2 - 2ac + 2c^2)c\\
&= \varphi^3(q)\varphi^2(q^2)\varphi(q^4)\\
&= \sum_{n=0}^{\infty} N(1, 1, 1, 2, 2, 4; n)q^n.
\end{aligned}
$$

Equating coefficients of q^n $(n \in \mathbb{N})$, we obtain the asserted formula for $N(1,1,1,2,2,4;n)$. $\square$

Proof of (vii). By Theorems 2.7, 2.2(i), 2.1 and Eqs. (3.1)–(3.3), (3.5), we have

$$\begin{aligned}
1 - 4 \sum_{\substack{n=1 \\ n \equiv 0 \pmod 4}}^{\infty} H_4(n)q^n + 2\sum_{n=1}^{\infty} G_4(n)q^n + \sum_{\substack{n=1 \\ n \equiv 1 \pmod 4}}^{\infty} I(n)q^n \\
&= \varphi^2(q^4)\varphi^4(-q^4) + \frac{1}{8}\varphi^6(q) - \frac{1}{8}\varphi^2(q)\varphi^4(-q) \\
&\quad + \frac{1}{8}\varphi^5(q)\varphi(-q) - \frac{1}{8}\varphi(q)\varphi^5(-q) \\
&= c^2(a^2 - 2ac + 2c^2)(2ac - a^2) + \frac{1}{8}a^6 - \frac{1}{8}a^2(2c-a)^4 \\
&\quad + \frac{1}{8}a^5(2c-a) - \frac{1}{8}a(2c-a)^5 \\
&= a^2(a^2 - 2ac + 2c^2)c^2 \\
&= \varphi^2(q)\varphi^2(q^2)\varphi^2(q^4) \\
&= \sum_{n=0}^{\infty} N(1,1,2,2,4,4;n)q^n.
\end{aligned}$$

Equating coefficients of q^n $(n \in \mathbb{N})$, we obtain the asserted formula for $N(1,1,2,2,4,4;n)$. $\square$

Proof of (viii). By Theorems 2.7, 2.2(i), 2.1 and Eqs. (3.1)–(3.3), (3.5), we have

$$\begin{aligned}
1 - 4 \sum_{\substack{n=1 \\ n \equiv 0 \pmod 4}}^{\infty} H_4(n)q^n + \sum_{n=1}^{\infty} G_4(n)q^n + \frac{1}{2}\sum_{\substack{n=1 \\ n \equiv 1 \pmod 4}}^{\infty} I(n)q^n \\
&= \varphi^2(q^4)\varphi^4(-q^4) + \frac{1}{16}\varphi^6(q) - \frac{1}{16}\varphi^2(q)\varphi^4(-q) \\
&\quad + \frac{1}{16}\varphi^5(q)\varphi(-q) - \frac{1}{16}\varphi(q)\varphi^5(-q) \\
&= c^2(a^2 - 2ac + 2c^2)(2ac - a^2) + \frac{1}{16}a^6 - \frac{1}{16}a^2(2c-a)^4 \\
&\quad + \frac{1}{16}a^5(2c-a) - \frac{1}{16}a(2c-a)^5 \\
&= a(a^2 - 2ac + 2c^2)c^3
\end{aligned}$$

$$= \varphi(q)\varphi^2(q^2)\varphi^3(q^4)$$
$$= \sum_{n=0}^{\infty} N(1,2,2,4,4,4;n)q^n.$$

Equating coefficients of q^n $(n \in \mathbb{N})$, we obtain the asserted formula for $N(1,2,2,4,4,4;n)$. □

4. Conclusion

We conclude this paper by showing how to give an alternative formulation of Theorem 1.1. Let $n \in \mathbb{N}$. We define $\alpha \in \mathbb{N}_0$ and $N \in \mathbb{N}$ with $N \equiv 1 \pmod 2$ uniquely by $n = 2^\alpha N$. Then

$$G_4(n) = 2^{2\alpha}G_4(N), \quad H_4(n) = H_4(N)$$

and

$$H_4(N) = \left(\frac{-4}{N}\right) G_4(N) = (-1)^{(N-1)/2}G_4(N).$$

Thus, by Theorem 1.1(i), we have

$$N(1,1,1,1,1,4;n)$$
$$= \begin{cases} 6G_4(N) + 2I(N), & \text{if } \alpha = 0, \quad N \equiv 1 \pmod 4, \\ 10G_4(N), & \text{if } \alpha = 0, \quad N \equiv 3 \pmod 4, \\ 40G_4(N), & \text{if } \alpha = 1, \\ 4(2^{2\alpha-1}\cdot 3 - (-1)^{(N-1)/2})G_4(N), & \text{if } \alpha \geq 2. \end{cases}$$

In a similar manner, we can express the remaining formulae of Theorem 1.1 in terms of α, N, $G_4(N)$ and $I(N)$.

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