FOURTEEN OCTONARY QUADRATIC FORMS

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We use the recent evaluation of certain convolution sums involving the sum of divisors function to determine the number of representations of a positive integer by certain diagonal octonary quadratic forms whose coefficients are 1, 2 or 4.

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1. Introduction
Let \( \mathbb{N} \) denote the set of positive integers and \( \mathbb{Z} \) the set of all integers. Set \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). For \( k \in \mathbb{N} \) we set
\[
σ_k(n) := \begin{cases} \sum_{d \in \mathbb{N}} d^k, & \text{if } n \in \mathbb{N}, \\ 0, & \text{if } n \notin \mathbb{N}. \end{cases} \tag{1.1}
\]
We write \( σ(n) \) for \( σ_1(n) \).

In this paper, we use the evaluation of the convolution sums
\[
\sum_{m \in \mathbb{N}} σ(m)σ(n - km)
\]
for \( k \in \{1, 2, 4, 8, 16\} \) to determine the number of representations \((x_1, \ldots, x_8) \in \mathbb{Z}^8\) of the positive integer \( n \) by each of the 14 diagonal octonary quadratic forms
\[
\begin{align*}
[800] & \quad x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 + x_8^2, \\
[620] & \quad x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + 2x_7^2 + 2x_8^2.
\end{align*}
\]
We write \( n \) gave a formula for the number of representations of \( n \) and determine notation \([11]\). Williams \([12, Theorem 2, p. 388]\) has determined Theorem 1.1. Alaca, Alaca and Williams \([2, Theorem 1.2, p. 4]\) have determined (iii) \( n \), (ii) \( n \), (i) \( n \), \( n \), \( n \), \( n \), \( n \), \( n \), \( n \), \( n \), \( n \), \( n \), \( n \), \( n \), \( n \), \( n \), \( n \), \( n \). (We found the adjective “octonary” in the dictionary \([6, Vol. 2, p. 914]\).) The notation \([abc] (a, b, c \in \mathbb{N}, a + b + c = 8)\) identifies the diagonal octonary quadratic form \( A_1x_1^2 + \cdots + A_8x_8^2 \) as having

\[ A_1 = \cdots = A_8 = 1, \]

\[ A_{a+1} = \cdots = A_{a+b} = 2, \]

\[ A_{a+b+1} = \cdots = A_{8} = 4. \]

We write \( N([abc]; n) \) for the number of representations of \( n \) \((n \in \mathbb{N})\) by the form \([abc]\), and determine \( N([abc]; n) \) for the 14 forms \([800], [620], [521], [440], [422], [404], [341], [323], [260], [242], [224], [161], [143] \) and \([125]\). The number \( N([800]; n) \) is just the number of representations of \( n \) as the sum of eight squares. Jacobi implicitly gave a formula for \( N([800]; n) \) in his famous work on elliptic functions \([9, §§40–42, pp. 159–170]\). An arithmetic proof of Jacobi’s eight squares formula has been given in \([11]\). Williams \([12, Theorem 2, p. 388]\) has determined \( N([440]; n) \) and Alaca, Alaca and Williams \([2, Theorem 1.2, p. 4]\) have determined \( N([404]; n) \). The evaluation of \( N([abc]; n) \) for the remaining 11 forms \([abc]\) is new.

**Theorem 1.1.** Let \( n \in \mathbb{N} \). Then

(i) \( N([800]; n) = 16\sigma_3(n) - 32\sigma_3(n/2) + 256\sigma_3(n/4), \)

(ii) \( N([620]; n) = 8\sigma_3(n) - 8\sigma_3(n/2) - 16\sigma_3(n/4) + 256\sigma_3(n/8) + 4c_8(n), \)

(iii) \( N([521]; n) = 4\sigma_3(n) - 4\sigma_3(n/2) - 16\sigma_3(n/4) + 256\sigma_3(n/16) \)

\[ + (2 + 4(\frac{-1}{n}))c_8(n) + 12c_8(n/2), \]

(iv) \( N([440]; n) = 4\sigma_3(n) - 4\sigma_3(n/2) - 16\sigma_3(n/4) + 256\sigma_3(n/8) + 4c_8(n), \)

(v) \( N([422]; n) = 2\sigma_3(n) - 2\sigma_3(n/2) - 16\sigma_3(n/4) + 256\sigma_3(n/16) \)

\[ + (2 + 4(\frac{-1}{n}))c_8(n) + 12c_8(n/2), \]
where the integers $c_8(n)$ are given by

$$\sum_{n=1}^{\infty} c_8(n)q^n = q \prod_{n=1}^{\infty} (1 - q^{2n})^4 (1 - q^{4n})^4, \quad (1.2)$$

and $(\frac{-4}{n})$ (n ∈ N) is the Legendre–Jacobi–Kronecker symbol for discriminant −4, that is

$$\left(\frac{-4}{n}\right) = \begin{cases} +1, & \text{if } n \equiv 1 \pmod{4}, \\ -1, & \text{if } n \equiv 3 \pmod{4}, \\ 0, & \text{if } n \equiv 0 \pmod{2}. \end{cases}$$

In [3], the quantity $N([abc]; n)$ is evaluated for $[abc] = [701], [602], [503], [404], [305], [206]$ and [107].

### 2. Some Convolution Sums

Let $n \in \mathbb{N}$. The formula

$$\sum_{m \in \mathbb{N}}^{m < n} \sigma(m)\sigma(n-m) = \frac{5}{12} \sigma_3(n) + \left(\frac{1}{12} - \frac{1}{2^n}\right) \sigma(n) \quad (2.1)$$

originally appeared in a letter from Besge to Liouville [5]. Lützen [10, p. 81] indicates that Besge is a pseudonym for Liouville. Many proofs of (2.1) have been given, see for example [7]. An arithmetic proof is given in [8, p. 236]. The following two
evaluations are due to Huard, Ou, Spearman and Williams [8, Theorem 2, p. 247; Theorem 4, p. 249]

\[
\sum_{m \in \mathbb{N}, m < n/2} \sigma(m)\sigma(n-2m) = \frac{1}{12}\sigma_3(n) + \frac{1}{3}\sigma_3(n/2) + \left(\frac{1}{24} - \frac{1}{8}\right)\sigma(n) \\
+ \left(\frac{1}{24} - \frac{1}{4}\right)\sigma(n/2)
\] (2.2)

and

\[
\sum_{m \in \mathbb{N}, m < n/4} \sigma(m)\sigma(n-4m) = \frac{1}{48}\sigma_3(n) + \frac{1}{16}\sigma_3(n/2) + \frac{1}{3}\sigma_3(n/4) \\
+ \left(\frac{1}{24} - \frac{1}{16}\right)\sigma(n) + \left(\frac{1}{24} - \frac{1}{4}\right)\sigma(n/4)
\] (2.3)

Recently Williams [12] has shown that

\[
\sum_{m \in \mathbb{N}, m < n/8} \sigma(m)\sigma(n-8m) = \frac{1}{192}\sigma_3(n) + \frac{1}{64}\sigma_3(n/2) + \frac{1}{16}\sigma_3(n/4) + \frac{1}{3}\sigma_3(n/8) \\
+ \left(\frac{1}{24} - \frac{1}{32}\right)\sigma(n) + \left(\frac{1}{24} - \frac{1}{4}\right)\sigma(n/8) - \frac{1}{64}c_8(n),
\] (2.4)

where the integers \(c_8(n)\) \((n \in \mathbb{N})\) are defined in (1.2). Clearly

\[
c_8(n) = 0, \quad \text{if } n \equiv 0 \pmod{2},
\] (2.5)

Recently Alaca, Alaca and Williams [2] have proved that

\[
\sum_{m \in \mathbb{N}, m < n/16} \sigma(m)\sigma(n-16m) \\
= \frac{1}{768}\sigma_3(n) + \frac{1}{256}\sigma_3(n/2) + \frac{1}{64}\sigma_3(n/4) + \frac{1}{16}\sigma_3(n/8) + \frac{1}{3}\sigma_3(n/16) \\
+ \left(\frac{1}{24} - \frac{1}{64}\right)\sigma(n) + \left(\frac{1}{24} - \frac{1}{4}\right)\sigma(n/16) - \frac{7}{256}c_{16}(n),
\] (2.6)

where the rational numbers \(c_{16}(n)\) \((n \in \mathbb{N})\) are defined by

\[
\sum_{n=1}^{\infty} c_{16}(n)q^n = \frac{3}{32}A_1(q) + \frac{3}{112}A_2(q) + \frac{1}{224}A_3(q) \\
- \frac{1}{32}A_5(q) - \frac{3}{112}A_6(q) - \frac{1}{224}A_7(q),
\] (2.7)

where

\[
A_k(q) := \prod_{n=1}^{\infty} \left(1 + q^n\right)^{24-4k}\left(1 - q^n\right)^8\left(1 - q^{4n-2}\right)^{16-2k}.
\] (2.8)
More recently the authors [3] have proved that

$$c_{16}(n) = \left( \frac{3}{7} + \frac{4}{7} \left( \frac{-4}{n} \right) \right) c_8(n) + \frac{12}{7} c_8(n/2). \quad (2.9)$$

Thus (2.6) becomes

$$\sum_{\substack{m \in \mathbb{N} \atop m < n/16}} \sigma(m) \sigma(n - 16m) = \frac{1}{768} \sigma_3(n) + \frac{1}{256} \sigma_3(n/2) + \frac{1}{64} \sigma_3(n/4)$$

$$+ \frac{1}{16} \sigma_3(n/8) + \frac{1}{3} \sigma_3(n/16) + \left( \frac{1}{24} - \frac{1}{64} n \right) \sigma(n)$$

$$+ \left( \frac{1}{24} - \frac{1}{4} n \right) \sigma(n/16) - \left( \frac{3}{256} + \frac{1}{64} \left( \frac{-4}{n} \right) \right) c_8(n)$$

$$- \frac{3}{64} c_8(n/2). \quad (2.10)$$

We require the following result, which is a simple consequence of (2.1)–(2.4) and (2.10).

**Theorem 2.1.** Let $n \in \mathbb{N}$. Let $k \in \{1, 2, 4\}$. Let $\beta \in \{0, 1, 2, 3, 4\}$ and

- $\alpha \in \{0, 1, 2, 3, 4\}$, if $k = 1$,
- $\alpha \in \{0, 1, 2, 3\}$, if $k = 2$,
- $\alpha \in \{0, 1, 2\}$, if $k = 4$.

Then there exist 18 rational numbers

$$A_0, A_1, A_2, A_3, A_4, B_0, B_1, B_2, B_3, B_4, C_0, C_1, C_2, C_3, C_4, D_0, D_1, E,$$

which depend upon $k$, $\alpha$ and $\beta$ but not on $n$, such that

$$\sum_{\substack{m \in \mathbb{N} \atop m < n/k}} \sigma(m/2^\alpha) \sigma((n - km)/2^\beta) = \sum_{r=0}^{4} (A_r \sigma_3(n/2^r) + (B_r + C_r n) \sigma(n/2^r))$$

$$+ \left( D_0 + D_1 \left( \frac{-4}{n} \right) \right) c_8(n) + Ec_8(n/2).$$

**Proof.** It suffices to treat one case as the remaining cases can be treated in a similar manner.
Suppose $k = 1$ and $(\alpha, \beta) = (1, 4)$. Then
\[
\sum_{m \in \mathbb{N}} \sigma(m/2) \sigma((n - m)/16)
\]
\[
= \sum_{m \in \mathbb{N}} \sigma(m) \sigma((n - 2m)/16)
\]
\[
= \sum_{l \in \mathbb{N}} \sigma(l) \sigma(m)
\]
\[
= \sum_{l \in \mathbb{N}} \sigma(l) \sigma \left( \frac{n}{2} - 8l \right)
\]
\[
= \frac{1}{192} \sigma_3(n/2) + \frac{1}{64} \sigma_3(n/4) + \frac{1}{16} \sigma_3(n/8) + \frac{1}{3} \sigma_3(n/16)
\]
\[
+ \left( \frac{1}{24} - \frac{1}{64} n \right) \sigma(n/2) + \left( \frac{1}{24} - \frac{1}{8} n \right) \sigma(n/16) - \frac{1}{64} \epsilon_8(n/2),
\]
by (2.4). This is of the asserted form.

3. Three Quaternary Forms

We define three diagonal quaternary quadratic forms $f_1$, $f_2$, $f_3$ by
\[
f_1(x_1, x_2, x_3, x_4) := x_1^2 + x_2^2 + x_3^2 + x_4^2,
\]
\[
f_2(x_1, x_2, x_3, x_4) := x_1^2 + x_2^2 + 2x_3^2 + 2x_4^2,
\]
\[
f_3(x_1, x_2, x_3, x_4) := x_1^2 + 2x_2^2 + 2x_3^2 + 4x_4^2.
\]

It is known that
\[
N_{f_1}(n) := \text{card}\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid n = f_1(x_1, x_2, x_3, x_4)\}
\]
\[
= 8\sigma(n) - 32\sigma(n/4), \quad n \in \mathbb{N},
\]
\[
N_{f_2}(n) := \text{card}\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid n = f_2(x_1, x_2, x_3, x_4)\}
\]
\[
= 4\sigma(n) - 4\sigma(n/2) + 8\sigma(n/4) - 32\sigma(n/8), \quad n \in \mathbb{N},
\]
\[
N_{f_3}(n) := \text{card}\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid n = f_3(x_1, x_2, x_3, x_4)\}
\]
\[
= 2\sigma(n) - 2\sigma(n/2) + 8\sigma(n/8) - 32\sigma(n/16), \quad n \in \mathbb{N},
\]
see for example [1, pp. 296, 297, 300]. Hence, for $n \in \mathbb{N}$ and $i \in \{1, 2, 3\}$, we have
\[
N_{f_i}(n) = \sum_{\alpha=0}^{i+1} c_i(\alpha) \sigma(n/2^\alpha)
\]
for rational numbers $c_i(\alpha)$, which depend on $i$ and $\alpha$ but not on $n$. 
We observe that
\[ f_1(x_1, x_2, x_3, x_4) + f_1(x_5, x_6, x_7, x_8) = [800], \]
\[ f_1(x_1, x_2, x_3, x_4) + f_2(x_5, x_6, x_7, x_8) = [620], \]
\[ f_1(x_1, x_2, x_3, x_4) + f_3(x_5, x_6, x_7, x_8) = [521], \]
\[ f_2(x_1, x_2, x_3, x_4) + f_2(x_5, x_6, x_7, x_8) = [440], \]
\[ f_2(x_1, x_2, x_3, x_4) + f_3(x_5, x_6, x_7, x_8) = [341], \]
\[ f_3(x_1, x_2, x_3, x_4) + f_3(x_5, x_6, x_7, x_8) = [242]. \]

Also
\[ f_1(x_1, x_2, x_3, x_4) + 2f_2(x_5, x_6, x_7, x_8) = [422], \]
\[ f_2(x_1, x_2, x_3, x_4) + 2f_1(x_5, x_6, x_7, x_8) = [260], \]
\[ f_3(x_1, x_2, x_3, x_4) + 2f_1(x_5, x_6, x_7, x_8) = [161], \]
\[ f_3(x_1, x_2, x_3, x_4) + 2f_2(x_5, x_6, x_7, x_8) = [143]. \]

Finally
\[ f_1(x_1, x_2, x_3, x_4) + 4f_1(x_5, x_6, x_7, x_8) = [404], \]
\[ f_2(x_1, x_2, x_3, x_4) + 4f_1(x_5, x_6, x_7, x_8) = [224], \]
\[ f_3(x_1, x_2, x_3, x_4) + 4f_1(x_5, x_6, x_7, x_8) = [125]. \]

We note that
\[ [323] \neq f_1(x_1, x_2, x_3, x_4) + rf_3(x_5, x_6, x_7, x_8), \quad i, j \in \{1, 2, 3\}, \quad r \in \{1, 2, 4\}. \]

For this reason the determination of \( N([323]; n) \ (n \in \mathbb{N}) \) is handled separately and differently from the others. Combinations \( f_i(x_1, x_2, x_3, x_4) + rf_j(x_5, x_6, x_7, x_8) \) not listed above either lead to repetitions (for example \( f_1(x_1, x_2, x_3, x_4) + 2f_1(x_5, x_6, x_7, x_8) = [440] \)) or to parameters outside the range of applicability of Theorem 2.1 (for example the proof in Sec. 5 applied to \( f_1(x_1, x_2, x_3, x_4) + 2f_3(x_5, x_6, x_7, x_8) \) allows \( \alpha = 4, \beta = 2 \) and Theorem 2.1 with \( k = 2 \) is not applicable).

4. The Forms \([800], [620], [521], [440], [341] \) and \([242]\)

For \( n \in \mathbb{N} \) and \( i, j \in \{1, 2, 3\} \) with \( i \leq j \), we have by (3.7)
\[
N_{f_i + f_j}(n) := \text{card} \{(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \in \mathbb{Z}^8 \mid \ n = f_i(x_1, x_2, x_3, x_4) + f_j(x_5, x_6, x_7, x_8)\}
\]
\[
= N_{f_i}(n) + N_{f_j}(n) + \sum_{m=1}^{n-1} N_{f_i}(n - m) N_{f_j}(m)
\]
following values of \( i, j \)
\[
A \quad \text{for some rational numbers} \quad \{ \alpha, \beta \} \in \mathbb{Q}.
\]
For \((i, j) = (1, 2, 3)\) we have \(\alpha, \beta \in \{0, 1, 2, 3, 4\}\). Thus, by Theorem 2.1 with \( k = 1 \), we have
\[
N_{f_1+f_j}(n) = A_0\sigma_3(n) + A_1\sigma_3(n/2) + A_2\sigma_3(n/4) + A_3\sigma_3(n/8) + A_4\sigma_3(n/16)
+ (B_0 + C_0n)\sigma(n) + (B_1 + C_1n)\sigma(n/2) + (B_2 + C_2n)\sigma(n/4)
+ (B_3 + C_3n)\sigma(n/8) + (B_4 + C_4n)\sigma(n/16)
+ \left(D_0 + D_1 \left(\frac{4}{n}\right)\right)c_8(n) + Ec_8(n/2)
\]
for some rational numbers \(A_0, A_1, \ldots, E\) depending on \(i\) and \(j\) but not on \(n\). Next, determining \(N_{f_1+f_j}(n)\) numerically for \(n = 1, 2, \ldots, 18\) for each pair \((i, j)\) with \(i, j \in \{1, 2, 3\}\) and \(i \leq j\), we obtain 18 linearly independent linear equations for the 18 quantities \(A_0, A_1, \ldots, E\). Using MAPLE to solve these equations, we obtain the following values of \(A_0, A_1, \ldots, E\). For each of the six pairs \((i, j)\) we find
\[
B_0 = B_1 = B_2 = B_3 = B_4 = C_0 = C_1 = C_2 = C_3 = C_4 = 0.
\]
For \((i, j) = (1, 1)\), so that \(f_1 + f_j = [800]\), we find
\[
A_0 = 16, \quad A_1 = -32, \quad A_2 = 256, \quad A_3 = 0, \quad A_4 = 0,
D_0 = 0, \quad D_1 = 0, \quad E = 0.
\]
For \((i, j) = (1, 2)\), so that \(f_1 + f_j = [620]\), we find
\[
A_0 = 8, \quad A_1 = -8, \quad A_2 = -16, \quad A_3 = 256, \quad A_4 = 0,
D_0 = 4, \quad D_1 = 0, \quad E = 0.
\]
For \((i, j) = (1, 3)\), so that \(f_1 + f_j = [521]\), we find
\[
A_0 = 4, \quad A_1 = -4, \quad A_2 = 0, \quad A_3 = -16, \quad A_4 = 256,
D_0 = 2, \quad D_1 = 4, \quad E = 12.
\]
For \((i, j) = (2, 2)\), so that \(f_1 + f_j = [440]\), we find
\[
A_0 = 4, \quad A_1 = -4, \quad A_2 = -16, \quad A_3 = 256, \quad A_4 = 0,
D_0 = 4, \quad D_1 = 0, \quad E = 0.
\]
For \((i, j) = (2, 3)\), so that \(f_1 + f_j = [341]\), we find
\[
A_0 = 2, \quad A_1 = -2, \quad A_2 = 0, \quad A_3 = -16, \quad A_4 = 256,
D_0 = 2, \quad D_1 = 2, \quad E = 4.
\]
For \((i, j) = (3, 3)\), so that \(f_i + f_j = [242]\), we find
\[
A_0 = 1, \quad A_1 = -1, \quad A_2 = 0, \quad A_3 = -16, \quad A_4 = 256,
\]
\[
D_0 = 1, \quad D_1 = 2, \quad E = 4.
\]
This completes the proof of parts (i)–(iv), (vii) and (x) of Theorem 1.1.

5. The Forms \([422], [260], [161] \text{ and } [143]\)

For \(n \in \mathbb{N}\) and \((i, j) = (1, 2), (2, 1), (3, 1) \text{ and } (3, 2)\), we have by (3.7)
\[
N_{f_i + 2f_j}(n) := \text{card}\{(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \in \mathbb{Z}^8 | n = f_i(x_1, x_2, x_3, x_4) + 2f_j(x_5, x_6, x_7, x_8)\}
\]
\[
= N_{f_i}(n) + N_{f_j}(n/2) + \sum_{m \in \mathbb{N}} N_{f_i}(n - 2m)N_{f_j}(m)
\]
\[
= \sum_{\alpha = 0}^{j+1} c_i(\alpha)\sigma(n/2^\alpha) + \sum_{\alpha = 0}^{i+1} c_j(\alpha)\sigma(n/2^{\alpha + 1})
\]
\[
+ \sum_{\alpha = 0}^{j+1} \sum_{\beta = 0}^{i+1} c_j(\alpha)c_i(\beta) \sum_{m \in \mathbb{N}} \sum_{m < n/2} \sigma(m/2^\alpha)\sigma((n - 2m)/2^\beta).
\]

For the pairs \((i, j)\) under consideration we have
\[
0 \leq \alpha \leq j + 1 \leq 3
\]
and
\[
0 \leq \beta \leq i + 1 \leq 4
\]
so, by Theorem 2.1 with \(k = 2\), we have
\[
N_{f_i + 2f_j}(n) = A_0\sigma_3(n) + A_1\sigma_3(n/2) + A_2\sigma_3(n/4) + A_3\sigma_3(n/8) + A_4\sigma_3(n/16)
\]
\[
+ (B_0 + C_0n)\sigma(n) + (B_1 + C_1n)\sigma(n/2) + (B_2 + C_2n)\sigma(n/4)
\]
\[
+ (B_3 + C_3n)\sigma(n/8) + (B_4 + C_4n)\sigma(n/16)
\]
\[
+ \left( D_0 + D_1 \left( -\frac{4}{n} \right) \right) c_8(n) + Ec_8(n/2).
\]

Next, determining \(N_{f_i + 2f_j}(n)\) numerically for \(n = 1, 2, \ldots, 18\) for each of the four specified pairs \((i, j)\), we obtain 18 linearly independent linear equations for the 18 quantities \(A_0, A_1, \ldots, E\). Using MAPLE to solve these equations, we find that for all four pairs \((i, j)\)
\[
B_0 = B_1 = B_2 = B_3 = B_4 = C_0 = C_1 = C_2 = C_3 = C_4 = 0.
\]
The values of $A_0, \ldots, A_4, D_0, D_1$ and $E$ are given below. For $(i, j) = (1, 2)$, so that $f_i + 2f_j = [422]$, we find

$$A_0 = 2, \quad A_1 = -2, \quad A_2 = 0, \quad A_3 = -16, \quad A_4 = 256,$$

$$D_0 = 2, \quad D_1 = 4, \quad E = 12.$$

For $(i, j) = (2, 1)$, so that $f_i + 2f_j = [260]$, we find

$$A_0 = 2, \quad A_1 = -2, \quad A_2 = 0, \quad A_3 = -16, \quad A_4 = 0,$$

$$D_0 = 2, \quad D_1 = 0, \quad E = 0.$$

For $(i, j) = (3, 1)$, so that $f_i + 2f_j = [161]$, we find

$$A_0 = 1, \quad A_1 = -1, \quad A_2 = 0, \quad A_3 = -16, \quad A_4 = 256,$$

$$D_0 = 1, \quad D_1 = 0, \quad E = 4.$$

For $(i, j) = (3, 2)$, so that $f_i + 2f_j = [143]$, we find

$$A_0 = \frac{1}{2}, \quad A_1 = -\frac{1}{2}, \quad A_2 = 0, \quad A_3 = -16, \quad A_4 = 256,$$

$$D_0 = \frac{1}{2}, \quad D_1 = 1, \quad E = 4.$$

This completes the proof of parts (v), (ix), (xii) and (xiii).

\[\square\]

### 6. The Forms [404], [224] and [125]

For $n \in \mathbb{N}$ and $(i, j) = (1,1), (2,1)$ and $(3,1)$, we have by (3.7)

$$N_{f_i+4f_j}(n) := \text{card}\{(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \in \mathbb{Z}^8 |$$

$$n = f_i(x_1, x_2, x_3, x_4) + 4f_j(x_5, x_6, x_7, x_8)\}$$

$$= N_{f_i}(n) + N_{f_j}(n/4) + \sum_{m \in \mathbb{N}} N_{f_i}(n-4m)N_{f_j}(m)$$

$$= \sum_{\alpha=0}^{i+1} c_i(\alpha)\sigma(n/2^\alpha) + \sum_{\alpha=0}^{j+1} c_j(\alpha)\sigma(n/2^{\alpha+2})$$

$$+ \sum_{\alpha=0}^{j+1} \sum_{\beta=0}^{i+1} c_i(\alpha)c_j(\beta) \sum_{m \in \mathbb{N}} \sigma(m/2^\alpha)\sigma((n-4m)/2^\beta).$$

For the pairs $(i, j)$ under consideration we have

$$0 \leq \alpha \leq j + 1 = 2$$

and

$$0 \leq \beta \leq i + 1 \leq 4$$
so, by Theorem 2.1 with \( k = 4 \), we have

\[
N_{f_i+4f_j}(n) = A_0 \sigma_3(n) + A_1 \sigma_3(n/2) + A_2 \sigma_3(n/4) + A_3 \sigma_3(n/8) + A_4 \sigma_3(n/16)
\]
\[
+ (B_0 + C_0 n) \sigma(n) + (B_1 + C_1 n) \sigma(n/2) + (B_2 + C_2 n) \sigma(n/4)
\]
\[
+ (B_3 + C_3 n) \sigma(n/8) + (B_4 + C_4 n) \sigma(n/16)
\]
\[
+ \left( D_0 + D_1 \left( \frac{-4}{n} \right) \right) c_8(n) + E c_8(n/2).
\]

Next, determining \( N_{f_i+4f_j}(n) \) numerically for \( n = 1, 2, \ldots, 18 \) for each of the three specified pairs \((i,j)\), we obtain 18 linearly independent linear equations for the 18 quantities \( A_0, A_1, \ldots, E \). Using MAPLE to solve these equations, we find that for all three pairs \((i,j)\)

\[
B_0 = B_1 = B_2 = B_3 = B_4 = C_0 = C_1 = C_2 = C_3 = C_4 = 0.
\]

The values of \( A_0, \ldots, A_4, D_0, D_1 \) and \( E \) are given below. For \((i,j) = (1,1)\), so that \( f_i + 4f_j = [404] \), we find

\[
A_0 = 1, \quad A_1 = 3, \quad A_2 = -68, \quad A_3 = 48, \quad A_4 = 256,
\]
\[
D_0 = 3, \quad D_1 = 4, \quad E = 12.
\]

For \((i,j) = (2,1)\), so that \( f_i + 4f_j = [224] \), we find

\[
A_0 = \frac{1}{2}, \quad A_1 = -\frac{1}{2}, \quad A_2 = 0, \quad A_3 = -16, \quad A_4 = 256,
\]
\[
D_0 = \frac{3}{2}, \quad D_1 = 2, \quad E = 4.
\]

For \((i,j) = (3,1)\), so that \( f_i + 4f_j = [125] \), we find

\[
A_0 = \frac{1}{4}, \quad A_1 = -\frac{1}{4}, \quad A_2 = 0, \quad A_3 = -16, \quad A_4 = 256,
\]
\[
D_0 = \frac{3}{4}, \quad D_1 = 1, \quad E = 2.
\]

This completes the proof of parts (vi), (xi) and (xiv). \( \square \)

7. The Form [323]

Let

\[
\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \quad q \in \mathbb{C}, \quad |q| < 1.
\]

It is well-known that [4, p. 71]

\[
\varphi(q) + \varphi(-q) = 2 \varphi(q^4)
\]

and [4, p. 72]

\[
\varphi^2(q) + \varphi^2(-q) = 2 \varphi^2(q^2).
\]
Hence
\[ \varphi^2(q) + (2\varphi(q^4) - \varphi(q))^2 = 2\varphi^2(q^2) \]
so that
\[ \varphi^2(q) - 2\varphi(q)\varphi(q^4) - \varphi^2(q^2) + 2\varphi^2(q^4) = 0. \]

Multiplying both sides by \( \varphi^r(q)\varphi^s(q^2)\varphi^{6-r-s}(q^4) \) \((r, s \in \mathbb{N_0}, r + s \leq 6)\), we obtain
\[
\varphi^{r+2}(q)\varphi^s(q^2)\varphi^{6-r-s}(q^4) - 2\varphi^{r+1}(q)\varphi^s(q^2)\varphi^{7-r-s}(q^4)
- \varphi^r(q)\varphi^{s+2}(q^2)\varphi^{6-r-s}(q^4) + 2\varphi^r(q)\varphi^s(q^2)\varphi^{8-r-s}(q^4) = 0.
\]

As
\[ \varphi^r(q)\varphi^s(q^2)\varphi^{8-t-u}(q^4) = \sum_{n=0}^{\infty} N([t u 8 - t - u]; n)q^n, \]
we deduce
\[
N([r + 2 s 6 - r - s]; n) - 2N([r + 1 s 7 - r - s]; n)
- N([r s + 26 - r - s]; n) + 2N([r s 8 - r - s]; n) = 0, n \in \mathbb{N}. \quad (7.1)
\]

Taking \( r = s = 2 \) in (7.1), we obtain
\[ N([422]; n) - 2N([323]; n) - N([242]; n) + 2N([224]; n) = 0. \]

Appealing to Theorem 1.1(v), (x), (xi), we deduce
\[
N([323]; n) = \frac{1}{2} N([422]; n) - \frac{1}{2} N([242]; n) + N([224]; n)
= \sigma_3(n) - \sigma_3(n/2) - 8\sigma_3(n/8) + 128\sigma_3(n/16)
+ \left(1 + 2 \left(\frac{-4}{n}\right)\right)c_8(n) + 6c_8(n/2)
- \frac{1}{2}\sigma_3(n) + \frac{1}{2}\sigma_3(n/2) + 8\sigma_3(n/8) - 128\sigma_3(n/16)
- \frac{1}{2}\left(1 + 2 \left(\frac{-4}{n}\right)\right)c_8(n) - 2c_8(n/2)
+ \frac{1}{2}\sigma_3(n) - \frac{1}{2}\sigma_3(n/2) - 16\sigma_3(n/8) + 256\sigma_3(n/16)
+ \left(\frac{3}{2} + 2 \left(\frac{-4}{n}\right)\right)c_8(n) + 4c_8(n/2)
= \sigma_3(n) - \sigma_3(n/2) - 16\sigma_3(n/8) + 256\sigma_3(n/16)
+ \left(2 + 3 \left(\frac{-4}{n}\right)\right)c_8(n) + 8c_8(n/2),
\]
which is part (viii) of Theorem 1.1. \(\square\)
The eight choices \((r, s) = (0, 2), (0, 4), (0, 6), (1, 2), (1, 4), (2, 4), (3, 2), (4, 2)\) in (7.1) lead to the relationships (as \(N([0ab]; n) = N([ab0]; n/2))\)

\[
\begin{align*}
N([224]; n) - 2N([125]; n) - N([440]; n/2) + 2N([260]; n/2) &= 0, \\
N([242]; n) - 2N([143]; n) - N([620]; n/2) + 2N([440]; n/2) &= 0, \\
N([260]; n) - 2N([161]; n) - N([800]; n/2) + 2N([620]; n/2) &= 0, \\
N([323]; n) - 2N([224]; n) - N([143]; n) + 2N([125]; n) &= 0, \\
N([341]; n) - 2N([242]; n) - N([161]; n) + 2N([143]; n) &= 0, \\
N([440]; n) - 2N([341]; n) - N([260]; n) + 2N([242]; n) &= 0, \\
N([521]; n) - 2N([422]; n) - N([341]; n) + 2N([323]; n) &= 0, \\
N([620]; n) - 2N([521]; n) - N([440]; n) + 2N([422]; n) &= 0.
\end{align*}
\]

These relations serve as checks on various parts of Theorem 1.1. The remaining choices of \((r, s)\) do not lead to new determinations of \(N([abc]; n)\).

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References


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