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EVALUATION OF THE SUMS $\sum_{\substack{m=1 \\ m\equiv a \pmod 4}}^{n-1} \sigma(m)\sigma(n-m)$

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Abstract. The convolution sum

$$\sum_{\substack{m=1 \\ m\equiv a \pmod 4}}^{n-1} \sigma(m)\sigma(n-m)$$

is evaluated for $a \in \{0,1,2,3\}$ and all $n \in \mathbb{N}$. This completes the partial evaluation given in the paper of J. G. Huard, Z. M. Ou, B. K. Spearman, K. S. Williams.

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1. Introduction

Let $\mathbb{N}$ denote the set of positive integers. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{Q}$ denote the set of rationals numbers. For $n \in \mathbb{N}$ and $k \in \mathbb{N}$ we set

$$\sigma_k(n) := \sum_{\substack{d\in\mathbb{N} \\ d\mid n}} d^k. \tag{1.1}$$

If $n \in \mathbb{Q}$ and $n \notin \mathbb{N}$, we set $\sigma_k(n) = 0$. We write $\sigma(n)$ for $\sigma_1(n)$. For $a \in \{0,1,2,3\}$ we define

$$S_{a,4}(n) := \sum_{\substack{m=1 \\ m\equiv a \pmod 4}}^{n-1} \sigma(m)\sigma(n-m). \tag{1.2}$$

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In [5, Theorem 9, p. 257] the authors gave a partial evaluation of the sums $S_{a,4}(n)$ ($a \in \{0,1,2,3\}$) using elementary considerations. They proved

$$S_{1,4}(n) = S_{3,4}(n) = \frac{1}{16}\sigma_3(n) - \frac{1}{16}\sigma_3(n/2), \quad \text{if } n \equiv 0 \pmod 4, \tag{1.3}$$

$$S_{0,4}(n) + S_{2,4}(n) = \frac{7}{24}\sigma_3(n) + \frac{1}{8}\sigma_3(n/2) + \Big(\frac{1}{12} - \frac{1}{2}n\Big)\sigma(n), \quad \text{if } n \equiv 0 \pmod 4, \tag{1.4}$$

$$S_{0,4}(n) = S_{1,4}(n), \quad \text{if } n \equiv 1 \pmod 4, \tag{1.5}$$

$$S_{2,4}(n) = S_{3,4}(n), \quad \text{if } n \equiv 1 \pmod 4, \tag{1.6}$$

$$S_{0,4}(n) + S_{2,4}(n) = \frac{5}{24}\sigma_3(n) + \Big(\frac{1}{24} - \frac{1}{4}n\Big)\sigma(n), \quad \text{if } n \equiv 1 \pmod 4, \tag{1.7}$$

$$S_{0,4}(n) = S_{2,4}(n) = \frac{11}{72}\sigma_3(n) + \Big(\frac{1}{24} - \frac{1}{4}n\Big)\sigma(n), \quad \text{if } n \equiv 2 \pmod 4, \tag{1.8}$$

$$S_{1,4}(n) + S_{3,4}(n) = \frac{1}{9}\sigma_3(n), \quad \text{if } n \equiv 2 \pmod 4, \tag{1.9}$$

$$S_{0,4}(n) = S_{3,4}(n), \quad \text{if } n \equiv 3 \pmod 4, \tag{1.10}$$

$$S_{1,4}(n) = S_{2,4}(n), \quad \text{if } n \equiv 3 \pmod 4, \tag{1.11}$$

$$S_{0,4}(n) + S_{1,4}(n) = \frac{5}{24}\sigma_3(n) + \Big(\frac{1}{24} - \frac{1}{4}n\Big)\sigma(n), \quad \text{if } n \equiv 3 \pmod 4. \tag{1.12}$$

In this paper we give a complete determination of the $S_{a,4}(n)$ ($a \in \{0,1,2,3\}$) valid for all $n \in \mathbb{N}$. We need the integers $c_8(n)$ ($n \in \mathbb{N}$) defined by

$$\sum_{n=1}^{\infty} c_8(n) q^n = q \prod_{n=1}^{\infty} (1-q^{2n})^4 (1-q^{4n})^4, \quad q \in \mathbb{C}, \ |q| < 1, \tag{1.13}$$

which were used in [6, Theorem 1, p. 388] to evaluate the convolution sum

$$\sum_{\substack{m \in \mathbb{N} \\ m < n/8}} \sigma(n)\sigma(n-8m).$$

(In [6] the integer $c_8(n)$ was denoted by $k(n)$.) Clearly

$$c_8(n) = 0, \quad \text{if } n \equiv 0 \pmod 2, \tag{1.14}$$

as noted in [6, p. 388]. We prove

Theorem 1.1. *Let $n \in \mathbb{N}$. If $n \equiv 0 \pmod 4$ then*

$$\begin{aligned}
S_{0,4}(n) &= \frac{29}{192}\sigma_3(n) + \frac{17}{64}\sigma_3(n/2) + \left(\frac{1}{12} - \frac{1}{2}n\right)\sigma(n),\\
S_{1,4}(n) &= \frac{1}{16}\sigma_3(n) - \frac{1}{16}\sigma_3(n/2),\\
S_{2,4}(n) &= \frac{9}{64}\sigma_3(n) - \frac{9}{64}\sigma_3(n/2),\\
S_{3,4}(n) &= \frac{1}{16}\sigma_3(n) - \frac{1}{16}\sigma_3(n/2).
\end{aligned}$$

If $n \equiv 1 \pmod 4$ then

$$\begin{aligned}
S_{0,4}(n) &= \frac{11}{96}\sigma_3(n) + \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma(n) + \frac{3}{32}c_8(n),\\
S_{1,4}(n) &= \frac{11}{96}\sigma_3(n) + \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma(n) + \frac{3}{32}c_8(n),\\
S_{2,4}(n) &= \frac{3}{32}\sigma_3(n) - \frac{3}{32}c_8(n),\\
S_{3,4}(n) &= \frac{3}{32}\sigma_3(n) - \frac{3}{32}c_8(n).
\end{aligned}$$

If $n \equiv 2 \pmod 4$, then

$$\begin{aligned}
S_{0,4}(n) &= \frac{11}{72}\sigma_3(n) + \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma(n),\\
S_{1,4}(n) &= \frac{1}{18}\sigma_3(n) + \frac{1}{2}c_8(n/2),\\
S_{2,4}(n) &= \frac{11}{72}\sigma_3(n) + \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma(n),\\
S_{3,4}(n) &= \frac{1}{18}\sigma_3(n) - \frac{1}{2}c_8(n/2).
\end{aligned}$$

If $n \equiv 3 \pmod 4$ then

$$\begin{aligned}
S_{0,4}(n) &= \frac{11}{96}\sigma_3(n) + \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma(n) + \frac{3}{32}c_8(n),\\
S_{1,4}(n) &= \frac{3}{32}\sigma_3(n) - \frac{3}{32}c_8(n),\\
S_{2,4}(n) &= \frac{3}{32}\sigma_3(n) - \frac{3}{32}c_8(n),\\
S_{3,4}(n) &= \frac{11}{96}\sigma_3(n) + \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma(n) + \frac{3}{32}c_8(n).
\end{aligned}$$

In view of (1.3)–(1.12), it suffices to determine $S_{0,4}(n)$ for $n \equiv 0, 1, 3 \pmod 4$ and $S_{1,4}(n)$ for $n \equiv 2 \pmod 4$, in order to complete the proof of Theorem 1.1. In

Section 2 we prove some results on theta functions that we shall need. In Section 3 we evaluate $S_{0,4}(n)$ for all $n \in \mathbb{N}$ and in Section 4 we evaluate $S_{1,4}(n)$ for all $n \in \mathbb{N}$ with $n \equiv 2 \pmod 4$.

2. Theta functions

Let q be a complex variable with $|q| < 1$. As in [2, p. 6] we set

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} \tag{2.1}$$

and

$$\psi(q) = \sum_{n=0}^{\infty} q^{\frac{1}{2}n(n+1)}. \tag{2.2}$$

The basic properties of φ and ψ are

$$\varphi(q) + \varphi(-q) = 2\varphi(q^4), \qquad [2, \text{Eq. } (3.6.1), \text{p. } 71], \tag{2.3}$$

$$\varphi(q) - \varphi(-q) = 4q\psi(q^8), \qquad [2, \text{Eq. } (3.6.2), \text{p. } 71], \tag{2.4}$$

$$\varphi(q)\psi(q^2) = \psi^2(q), \qquad [2, \text{Eq. } (3.6.3), \text{p. } 71], \tag{2.5}$$

$$\varphi^2(q) + \varphi^2(-q) = 2\varphi^2(q^2), \qquad [2, \text{Eq. } (3.6.7), \text{p. } 72], \tag{2.6}$$

$$\varphi^4(q) - \varphi^4(-q) = 16q\psi^4(q^2), \qquad [2, \text{Eq. } (3.6.8), \text{p. } 72], \tag{2.7}$$

$$\varphi(-q)\varphi(q) = \varphi^2(-q^2), \qquad [2, \text{Eq. } (1.3.32), \text{p. } 15]. \tag{2.8}$$

We need the following two identities.

Lemma 2.1. $\varphi^4(-q)\psi^4(q) + \varphi^4(q)\psi^4(-q) = 2\varphi^2(-q)\varphi^2(q^2)\psi^4(q)$.

Proof. We have

$$\begin{aligned}
&\varphi^4(-q)\psi^4(q) + \varphi^4(q)\psi^4(-q) \\
&\qquad = \varphi^4(-q)\varphi^2(q)\psi^2(q^2) + \varphi^4(q)\varphi^2(-q)\psi^2(q^2) \quad (\text{by } (2.5)) \\
&\qquad = \varphi^2(-q)\varphi^2(q)(\varphi^2(-q) + \varphi^2(q))\psi^2(q^2) \\
&\qquad = 2\varphi^2(-q)\varphi^2(q)\varphi^2(q^2)\psi^2(q^2) \quad (\text{by } (2.6)) \\
&\qquad = 2\varphi^2(-q)\varphi^2(q^2)\psi^4(q), \quad (\text{by } (2.5))
\end{aligned}$$

as asserted. □

Lemma 2.2. $\varphi^4(-q)\psi^4(q) - \varphi^4(q)\psi^4(-q) = -8q\varphi(q^2)\varphi^4(-q^2)\psi^3(q^4)$.

P r o o f. We have

$$\begin{aligned}
&\varphi^4(-q)\psi^4(q) - \varphi^4(q)\psi^4(-q)\\
&\qquad = \varphi^4(-q)\varphi^2(q)\psi^2(q^2) - \varphi^4(q)\varphi^2(-q)\psi^2(q^2) \quad \text{(by (2.5))}\\
&\qquad = \varphi^2(q)\varphi^2(-q)(\varphi^2(-q) - \varphi^2(q))\psi^2(q^2)\\
&\qquad = -8q\varphi^2(q)\varphi^2(-q)\varphi(q^4)\psi(q^8)\psi^2(q^2) \quad \text{(by (2.3) and (2.4))}\\
&\qquad = -8q\varphi^4(-q^2)\varphi(q^4)\psi(q^8)\psi^2(q^2) \quad \text{(by (2.8))}\\
&\qquad = -8q\varphi^4(-q^2)\psi^2(q^4)\psi^2(q^2) \quad \text{(by (2.5))}\\
&\qquad = -8q\varphi^4(-q^2)\psi^2(q^4)\varphi(q^2)\psi(q^4), \quad \text{(by (2.5))}\\
&\qquad = -8q\varphi(q^2)\varphi^4(-q^2)\psi^3(q^4),
\end{aligned}$$

as asserted. □

The infinite product representations of $\varphi(\pm q)$ and $\psi(\pm q)$ are due to Jacobi, namely,

$$\varphi(q) = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^5}{(1-q^n)^2(1-q^{4n})^2}, \quad \varphi(-q) = \prod_{n=1}^{\infty} \frac{(1-q^n)^2}{(1-q^{2n})}, \tag{2.9}$$

$$\psi(q) = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^2}{(1-q^n)}, \qquad \psi(-q) = \prod_{n=1}^{\infty} \frac{(1-q^n)(1-q^{4n})}{(1-q^{2n})}. \tag{2.10}$$

Lemma 2.3. $\displaystyle\sum_{\substack{n=1\\ n\equiv 1 \pmod 2}}^{\infty} c_8(n)q^n = q\varphi^4(-q^2)\psi^4(q^2)$.

P r o o f. We have

$$\begin{aligned}
q\varphi^4(-q^2)\psi^4(q^2) &= q\prod_{n=1}^{\infty}(1-q^{2n})^4(1-q^{4n})^4 \quad \text{(by (2.9) and (2.10))}\\
&= \sum_{\substack{n=1\\ n\equiv 1 \pmod 2}}^{\infty} c_8(n)q^n \quad \text{(by (1.13) and (1.14))}
\end{aligned}$$

as required. □

We are now ready to prove the main result of this section.

Theorem 2.1.

$$\text{(i)} \qquad \sum_{\substack{n=1\\ n\equiv 1 \pmod 4}}^{\infty} c_8(n)q^n = q\varphi^2(-q^2)\varphi^2(q^4)\psi^4(q^2);$$

$$\text{(ii)} \qquad \sum_{\substack{n=1\\ n\equiv 3 \pmod 4}}^{\infty} c_8(n)q^n = -4q^3\varphi(q^4)\varphi^4(-q^4)\psi^3(q^8).$$

P r o o f. (i) We have by Lemmas 2.3 and 2.1

$$\begin{aligned}
\sum_{\substack{n=1\\ n\equiv 1 \pmod 4}}^{\infty} c_8(n)q^n &= \sum_{\substack{n=1\\ n\equiv 1 \pmod 2}}^{\infty} c_8(n)\Big(\frac{2i+i^n-(-i)^n}{4i}\Big)q^n \\
&= \frac{1}{2}\sum_{\substack{n=1\\ n\equiv 1 \pmod 2}}^{\infty} c_8(n)q^n + \frac{1}{4i}\sum_{\substack{n=1\\ n\equiv 1 \pmod 2}}^{\infty} c_8(n)(iq)^n - \frac{1}{4i}\sum_{\substack{n=1\\ n\equiv 1 \pmod 2}}^{\infty} c_8(n)(-iq)^n \\
&= \frac{1}{2}q\varphi^4(-q^2)\psi^4(q^2) + \frac{1}{4i}(iq)\varphi^4(q^2)\psi^4(-q^2) - \frac{1}{4i}(-iq)\varphi^4(q^2)\psi^4(-q^2) \\
&= \frac{1}{2}q(\varphi^4(-q^2)\psi^4(q^2) + \varphi^4(q^2)\psi^4(-q^2)) \\
&= q\varphi^2(-q^2)\varphi^2(q^4)\psi^4(q^2).
\end{aligned}$$

(ii) We have by Lemmas 2.3 and 2.2

$$\begin{aligned}
\sum_{\substack{n=1\\ n\equiv 3 \pmod 4}}^{\infty} c_8(n)q^n &= \sum_{\substack{n=1\\ n\equiv 1 \pmod 2}}^{\infty} c_8(n)\Big(\frac{2i-i^n+(-i)^n}{4i}\Big)q^n \\
&= \frac{1}{2}\sum_{\substack{n=1\\ n\equiv 1 \pmod 2}}^{\infty} c_8(n)q^n - \frac{1}{4i}\sum_{\substack{n=1\\ n\equiv 1 \pmod 2}}^{\infty} c_8(n)(iq)^n + \frac{1}{4i}\sum_{\substack{n=1\\ n\equiv 1 \pmod 2}}^{\infty} c_8(n)(-iq)^n \\
&= \frac{1}{2}q\varphi^4(-q^2)\psi^4(q^2) - \frac{1}{4i}(iq)\varphi^4(q^2)\psi^4(-q^2) + \frac{1}{4i}(-iq)\varphi^4(q^2)\psi^4(-q^2) \\
&= \frac{1}{2}q(\varphi^4(-q^2)\psi^4(q^2) - \varphi^4(q^2)\psi^4(-q^2)) \\
&= \frac{1}{2}q(-8q^2\varphi(q^4)\varphi^4(-q^4)\psi^3(q^8)) \\
&= -4q^3\varphi(q^4)\varphi^4(-q^4)\psi^3(q^8).
\end{aligned}$$

□

Following Berndt [2, pp. 119–120] we set

$$x = 1 - \frac{\varphi^4(-q)}{\varphi^4(q)} \tag{2.11}$$

and

$$z = \varphi^2(q). \tag{2.12}$$

From Berndt's catalogue of formulae for theta functions [2, pp. 122–123], we have

$$\varphi(q) = \sqrt{z}, \tag{2.13}$$

$$\varphi(q^2) = \sqrt{z}\sqrt{\frac{1+\sqrt{1-x}}{2}}, \tag{2.14}$$

$$\varphi(q^4) = \frac{1}{2}\sqrt{z}(1+(1-x)^{1/4}), \tag{2.15}$$

$$\varphi(-q) = \sqrt{z}(1-x)^{1/4}, \tag{2.16}$$

$$\varphi(-q^2) = \sqrt{z}(1-x)^{1/8}, \tag{2.17}$$

$$\varphi(-q^4) = \sqrt{z}(1-x)^{1/16}\left(\frac{1+\sqrt{1-x}}{2}\right)^{1/4}, \tag{2.18}$$

$$\psi(q) = \sqrt{\frac{z}{2}}\left(\frac{x}{q}\right)^{1/8}, \tag{2.19}$$

$$\psi(q^2) = \frac{1}{2}\sqrt{z}\left(\frac{x}{q}\right)^{1/4}, \tag{2.20}$$

$$\psi(q^4) = \frac{1}{2}\sqrt{\frac{z}{2}}\left(\frac{1-\sqrt{1-x}}{q}\right)^{1/2}, \tag{2.21}$$

$$\psi(q^8) = \frac{1}{4}\sqrt{z}\,\frac{(1-(1-x)^{1/4})}{q}. \tag{2.22}$$

Appealing to these formulae and Theorem 2.1, we obtain

Theorem 2.2.

$$\text{(i)} \qquad \sum_{\substack{n=1 \\ n\equiv 1 \pmod 4}}^{\infty} c_8(n)q^n = \frac{1}{64}x(1-x)^{1/4}(1+(1-x)^{1/4})^2 z^4;$$

$$\text{(ii)} \qquad \sum_{\substack{n=1 \\ n\equiv 3 \pmod 4}}^{\infty} c_8(n)q^n = -\frac{1}{64}x(1-x)^{1/4}(1-(1-x)^{1/4})^2 z^4.$$

Following Cheng [3, p. 131] we set

$$g = (1-x)^{1/4}. \tag{2.23}$$

Then Theorem 2.2 can be reformulated as

Theorem 2.3.

(i) $$\sum_{\substack{n=1 \\ n\equiv 1 \pmod 4}}^{\infty} c_8(n)q^n = \frac{1}{64}(g+2g^2+g^3-g^5-2g^6-g^7)z^4;$$

(ii) $$\sum_{\substack{n=1 \\ n\equiv 3 \pmod 4}}^{\infty} c_8(n)q^n = \frac{1}{64}(-g+2g^2-g^3+g^5-2g^6+g^7)z^4.$$

We also need the sum $\sum\limits_{\substack{n=1 \\ n\equiv 2 \pmod 4}}^{\infty} c_8(n/2)q^n$ in terms of g and z.

Theorem 2.4.

$$\sum_{\substack{n=1 \\ n\equiv 2 \pmod 4}}^{\infty} c_8(n/2)q^n = \frac{1}{128}(g-g^3-g^5+g^7)z^4.$$

Proof. By Lemma 2.3, (2.18), (2.21) and (2.23), we have

$$\begin{aligned}\sum_{\substack{n=1 \\ n\equiv 2 \pmod 4}}^{\infty} c_8(n/2)q^n &= \sum_{\substack{n=1 \\ n\equiv 1 \pmod 2}}^{\infty} c_8(n)q^{2n} \\ &= q^2\varphi^4(-q^4)\psi^4(q^4) \\ &= \frac{1}{128}(1-x)^{1/4}(1+\sqrt{1-x})(1-\sqrt{1-x})^2 z^4 \\ &= \frac{1}{128}(g-g^3-g^5+g^7)z^4,\end{aligned}$$

as asserted. □

Let $\mathbb{Z}$ denote the set of integers. We define

$$L_{a,k}(q) := \sum_{\substack{n=1 \\ n\equiv a \pmod k}}^{\infty} \sigma(n)q^n, \quad a\in\mathbb{Z},\ k\in\mathbb{N}, \tag{2.24}$$

and

$$M_{a,k}(q) := \sum_{\substack{n=1 \\ n\equiv a \pmod{k}}}^{\infty} \sigma_3(n)q^n, \quad a \in \mathbb{Z},\ k \in \mathbb{N}. \tag{2.25}$$

The following two results are due to Cheng [3, Theorem 3.5.1, p. 139; Theorem 2.5.1, p. 67].

Theorem 2.5.

$$L_{1,4}(q) = \frac{1}{32}(1 + 2g - 2g^3 - g^4)z^2.$$

Theorem 2.6.

$$M_{1,2}(q) = \frac{1}{32}(1 - g^8)z^4.$$

We need $M_{1,2}(q^2)$ in terms of g and z.

Theorem 2.7.

$$M_{1,2}(q^2) = \frac{1}{512}(1 + 4g^2 - 10g^4 + 4g^6 + g^8)z^4.$$

Proof. Jacobi's duplication principle (see for example [2, Theorem 5.3.1, p. 121]) asserts that if $q \longrightarrow q^2$ then $x \longrightarrow \left((1-\sqrt{1-x})/(1+\sqrt{1-x})\right)^2$ and $z \longrightarrow \frac{1}{2}(1+\sqrt{1-x})z$. Thus

$$\begin{aligned} g^8 = (1-x)^2 &\longrightarrow \left(1 - \left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)^2\right)^2 \\ &= \left(1 - \left(\frac{1-g^2}{1+g^2}\right)^2\right)^2 = \frac{16g^4}{(1+g^2)^4} \end{aligned}$$

and

$$z \longrightarrow \frac{(1+g^2)}{2}z.$$

Hence, by Theorem 2.6, we obtain

$$\begin{aligned} M_{1,2}(q^2) &= \frac{1}{32}\left(1 - \frac{16g^4}{(1+g^2)^4}\right)\left(\frac{(1+g^2)}{2}z\right)^4 \\ &= \frac{1}{512}((1+g^2)^4 - 16g^4)z^4 \\ &= \frac{1}{512}(1 + 4g^2 - 10g^4 + 4g^6 + g^8)z^4 \end{aligned}$$

as asserted. □

3. Evaluation of $S_{0,4}(n)$ for all $n \in \mathbb{N}$

For any $m \in \mathbb{N}$ we have

$$\sigma(2m) = 3\sigma(m) - 2\sigma(m/2).$$

Thus

$$\begin{aligned}\sigma(4m) = 3\sigma(2m) - 2\sigma(m) &= 3(3\sigma(m) - 2\sigma(m/2)) - 2\sigma(m)\\ &= 7\sigma(m) - 6\sigma(m/2).\end{aligned}$$

Hence

$$\begin{aligned}S_{0,4}(n) = &\sum_{\substack{m=1\\ m\equiv 0 \pmod 4}}^{n-1} \sigma(m)\sigma(n-m)\\ &= \sum_{\substack{m\in\mathbb{N}\\ m<n/4}} \sigma(4m)\sigma(n-4m)\\ &= \sum_{\substack{m\in\mathbb{N}\\ m<n/4}} (7\sigma(m) - 6\sigma(m/2))\sigma(n-4m)\\ &= 7\sum_{\substack{m\in\mathbb{N}\\ m<n/4}} \sigma(m)\sigma(n-4m) - 6\sum_{\substack{m\in\mathbb{N}\\ m<n/8}} \sigma(m)\sigma(n-8m).\end{aligned}$$

It is shown in [5, Theorem 4, p. 249] that

$$\begin{aligned}\sum_{\substack{m\in\mathbb{N}\\ m<n/4}} \sigma(m)\sigma(n-4m) = &\frac{1}{48}\sigma_3(n) + \frac{1}{16}\sigma_3(n/2) + \frac{1}{3}\sigma_3(n/4)\\ &+ \left(\frac{1}{24} - \frac{1}{16}n\right)\sigma(n) + \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma(n/4)\end{aligned}$$

and in [6, Theorem 1, p. 388] that

$$\begin{aligned}\sum_{\substack{m\in\mathbb{N}\\ m<n/8}} \sigma(m)\sigma(n-8m) = &\frac{1}{192}\sigma_3(n) + \frac{1}{64}\sigma_3(n/2) + \frac{1}{16}\sigma_3(n/4) + \frac{1}{3}\sigma_3(n/8)\\ &+ \left(\frac{1}{24} - \frac{1}{32}n\right)\sigma(n) + \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma(n/8) - \frac{1}{64}c_8(n).\end{aligned}$$

Hence

$$S_{0,4}(n) = \frac{11}{96}\sigma_3(n) + \frac{11}{32}\sigma_3(n/2) + \frac{47}{24}\sigma_3(n/4) - 2\sigma_3(n/8) + \Big(\frac{1}{24} - \frac{1}{4}n\Big)\sigma(n)$$
$$+ \Big(\frac{7}{24} - \frac{7}{4}n\Big)\sigma(n/4) - \Big(\frac{1}{4} - \frac{3}{2}n\Big)\sigma(n/8) + \frac{3}{32}c_8(n).$$

If $n \equiv 1 \pmod 4$ or $n \equiv 3 \pmod 4$ we obtain

$$S_{0,4}(n) = \frac{11}{96}\sigma_3(n) + \Big(\frac{1}{24} - \frac{1}{4}n\Big)\sigma(n) + \frac{3}{32}c_8(n).$$

If $n \equiv 2 \pmod 4$ we obtain by (1.14)

$$\begin{aligned} S_{0,4}(n) &= \frac{11}{96}\sigma_3(n) + \frac{11}{32}\sigma_3(n/2) + \Big(\frac{1}{24} - \frac{1}{4}n\Big)\sigma(n) \\ &= \frac{11}{96}\sigma_3(n) + \frac{11}{288}\sigma_3(n) + \Big(\frac{1}{24} - \frac{1}{4}n\Big)\sigma(n) \\ &= \frac{11}{72}\sigma_3(n) + \Big(\frac{1}{24} - \frac{1}{4}n\Big)\sigma(n), \end{aligned}$$

as in (1.8). If $n \equiv 0 \pmod 4$ we obtain by (1.14)

$$\begin{aligned} S_{0,4}(n) &= \frac{11}{96}\sigma_3(n) + \frac{11}{32}\sigma_3(n/2) + \frac{47}{24}\Big(\frac{9}{8}\sigma_3(n/2) - \frac{1}{8}\sigma_3(n)\Big) \\ &\quad - 2\Big(\frac{73}{64}\sigma_3(n/2) - \frac{9}{64}\sigma_3(n)\Big) + \Big(\frac{1}{24} - \frac{1}{4}n\Big)\sigma(n) \\ &\quad + \Big(\frac{7}{24} - \frac{7}{4}n\Big)\Big(\frac{3}{2}\sigma(n/2) - \frac{1}{2}\sigma(n)\Big) \\ &\quad - \Big(\frac{1}{4} - \frac{3}{2}n\Big)\Big(\frac{7}{4}\sigma(n/2) - \frac{3}{4}\sigma(n)\Big) \\ &= \frac{29}{192}\sigma_3(n) + \frac{17}{64}\sigma_3(n/2) + \Big(\frac{1}{12} - \frac{1}{2}n\Big)\sigma(n). \end{aligned}$$

The formula for $S_{0,4}(n)$ when $n \equiv 0 \pmod 4$ is in agreement with that given in [4, Theorem 4.1, p. 570].

4. Evaluation of $S_{1,4}(n)$ for $n \equiv 2 \pmod 4$

By Theorem 2.4 we have

$$\begin{aligned}
\sum_{\substack{n=1\\ n\equiv 2 \ (\mathrm{mod}\ 4)}}^{\infty} S_{1,4}(n)q^n &= \sum_{n=1}^{\infty}\Bigg(\sum_{\substack{l,m\in\mathbb{N}\\ l+m=n\\ l\equiv m\equiv 1\ (\mathrm{mod}\ 4)}}^{\infty} \sigma(l)\sigma(m)\Bigg)q^n\\
&= \Bigg(\sum_{\substack{l=1\\ l\equiv 1\ (\mathrm{mod}\ 4)}}^{\infty} \sigma(l)q^l\Bigg)^2 = L_{1,4}^2(q)\\
&= \Big(\frac{1}{32}(1+2g-2g^3-g^4)z^2\Big)^2\\
&= \frac{1}{1024}(1+4g+4g^2-4g^3-10g^4-4g^5+4g^6+4g^7+g^8)z^4\\
&= \frac{1}{1024}(1+4g^2-10g^4+4g^6+g^8)z^4+\frac{1}{256}(g-g^3-g^5+g^7)z^4\\
&= \frac{1}{2}M_{1,2}(q^2)+\frac{1}{2}\sum_{\substack{n=1\\ n\equiv 2\ (\mathrm{mod}\ 4)}}^{\infty} c_8(n/2)q^n\\
&= \sum_{\substack{n=1\\ n\equiv 2\ (\mathrm{mod}\ 4)4}}^{\infty}\Big(\frac{1}{2}\sigma_3(n/2)+\frac{1}{2}c_8(n/2)\Big)q^n
\end{aligned}$$

so that

$$S_{1,4}(n) = \frac{1}{2}\sigma_3(n/2)+\frac{1}{2}c_8(n/2), \quad \text{if } n\equiv 2 \pmod 4.$$

5. Final remarks

The evaluations of Sections 3 and 4 complete the proof of Theorem 1.1. In the paper [1] the authors make use of Theorem 1.1 to determine the number of representations of a positive integer n by certain diagonal integral quadratic forms in eight variables.

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