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SOME NEW THETA FUNCTION IDENTITIES WITH APPLICATIONS TO SEXTENARY QUADRATIC FORMS

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Abstract

The (p, k)-parametrization of theta functions introduced by the authors is used to prove some new theta function identities from which formulae for the number of representations of a positive integer by each of the sextenary quadratic forms $x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+3x_6^2$, $x_1^2+x_2^2+x_3^2+3x_4^2+3x_5^2+3x_6^2$ and $x_1^2+3x_2^2+3x_3^2+3x_4^2+3x_5^2+3x_6^2$ are deduced.

Key Words and Phrases: (p, k)-parametrization of theta functions, representations by sextenary quadratic forms, theta function identities
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1. Introduction. Borwein, Borwein and Garvan, in their paper [6] on some cubic modular identities of Ramanujan, defined the functions

$$a(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}, \tag{1.1}$$

$$b(q) := \sum_{m,n=-\infty}^{\infty} \omega^{m-n} q^{m^2+mn+n^2}, \tag{1.2}$$

$$c(q) := \sum_{m,n=-\infty}^{\infty} q^{(m+\frac{1}{3})^2+(m+\frac{1}{3})(n+\frac{1}{3})+(n+\frac{1}{3})^2}, \tag{1.3}$$

where q is a complex variable satisfying $|q| < 1,\ \omega = e^{2\pi i/3}$, and in (1.3) the principal value of the cube root $q^{1/3}$ is taken. They proved a number of beautiful identities involving the functions $a,\ b$ and c, including [6, Theorem 2.3, p. 38]

$$a^3(q) = b^3(q) + c^3(q) \tag{1.4}$$

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and [6, Theorem 2.6, p. 40]

$$a(q)a(q^2) = b(q)b(q^2) + c(q)c(q^2). \tag{1.5}$$

In this paper we prove in Section 2 three new identities involving the functions b and c and the theta function φ (Ramanujan's notation [5, p. 6]) given by

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}. \tag{1.6}$$

Theorem 1..1.

(i) $\varphi^5(q)\varphi(q^3) = \frac{1}{9}b^3(-q) + \frac{8}{9}b^3(q^2) - \frac{1}{3}c^3(-q) + \frac{8}{3}c^3(q^2).$

(ii) $\varphi^3(q)\varphi^3(q^3) = \frac{1}{9}b^3(q) + \frac{8}{9}b^3(q^4) + \frac{1}{9}c^3(q) + \frac{8}{9}c^3(q^4) + 4q\prod_{n=1}^{\infty}(1-q^{2n})^3(1-q^{6n})^3.$

(iii) $\varphi(q)\varphi^5(q^3) = \frac{1}{9}b^3(-q) + \frac{8}{9}b^3(q^2) - \frac{1}{27}c^3(-q) + \frac{8}{27}c^3(q^2).$

We then use these three identities to give formulae for the number of representations of a positive integer by each of the three sextenary quadratic forms $x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+3x_6^2$, $x_1^2+x_2^2+x_3^2+3x_4^2+3x_5^2+3x_6^2$ and $x_1^2+3x_2^2+3x_3^2+3x_4^2+3x_5^2+3x_6^2$.

Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{Z}$ the set of all integers. Set $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$. For $a_1, a_2, a_3, a_4, a_5, a_6 \in \mathbb{N}$ and $n \in \mathbb{N}_0$ we set

$$N(a_1,a_2,a_3,a_4,a_5,a_6;n) = \text{card}\{(x_1,x_2,x_3,x_4,x_5,x_6) \in \mathbb{Z}^6 \mid n = a_1x_1^2+a_2x_2^2+a_3x_3^2+a_4x_4^2+a_5x_5^2+a_6x_6^2\}. \tag{1.7}$$

Clearly

$$N(a_1,a_2,a_3,a_4,a_5,a_6;0) = 1. \tag{1.8}$$

Also

$$\sum_{n=0}^{\infty} N(a_1,a_2,a_3,a_4,a_5,a_6;n)q^n = \prod_{r=1}^{6}\varphi(q^{a_r}). \tag{1.9}$$

We prove in Section 3

Theorem 1..2. *Let $n \in \mathbb{N}$. Set $n = 2^\alpha 3^\beta N$, where $\alpha \in \mathbb{N}_0$, $\beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and* $\gcd(N,6) = 1$. *Let $N = \prod_{p|N} p^{\alpha_p}$ be the prime factorization of N. Then*

(i) $N(1,1,1,1,1,3;n)$

$$= \frac{(-1)^{2^\alpha}}{5}(2^{2\alpha+2} - 9(-1)^\alpha)\Big(3^{2\beta+2} + (-1)^\alpha\Big(\frac{-3}{N}\Big)\Big)N^2$$

$$\times \prod_{p \mid N} \frac{1-\left(\frac{-3}{p}\right)^{\alpha_p+1} p^{-2\alpha_p-2}}{1-\left(\frac{-3}{p}\right)p^{-2}}.$$

(ii) $N(1,1,1,3,3,3;n)$

$$= \frac{1}{10}\Big(3(2^{2\alpha+1}+3(-1)^{\alpha})(1+(-1)^{2^{\alpha}})+5(1-(-1)^{2^{\alpha}})\Big)$$

$$\times\Big(3^{2\beta+1}-(-1)^{\alpha}\Big(\frac{-3}{N}\Big)\Big)N^2 \prod_{p \mid N} \frac{1-\left(\frac{-3}{p}\right)^{\alpha_p+1} p^{-2\alpha_p-2}}{1-\left(\frac{-3}{p}\right)p^{-2}} + 4k(n),$$

where the integers $k(n)$ $(n \in \mathbb{N})$ are defined by

$$q\prod_{n=1}^{\infty}(1-q^{2n})^3(1-q^{6n})^3 = \sum_{n=1}^{\infty} k(n)q^n. \tag{1.10}$$

(iii) $N(1,3,3,3,3,3;n)$

$$= \frac{(-1)^{2^{\alpha}}}{5}(2^{2\alpha+2}-9(-1)^{\alpha})\Big(3^{2\beta}+(-1)^{\alpha}\Big(\frac{-3}{N}\Big)\Big)N^2$$

$$\times \prod_{p \mid N} \frac{1-\left(\frac{-3}{p}\right)^{\alpha_p+1} p^{-2\alpha_p-2}}{1-\left(\frac{-3}{p}\right)p^{-2}}.$$

The first ten values of $k(n)$ are

$$k(1)=1,\ k(2)=0,\ k(3)=-3,\ k(4)=0,\ k(5)=0,$$
$$k(6)=0,\ k(7)=2,\ k(8)=0,\ k(9)=9,\ k(10)=0.$$

Clearly

$$k(n)=0, \text{ if } n \equiv 0 \pmod 2. \tag{1.11}$$

Liouville [11], [12] stated without proof formulae equivalent to (i) and (iii) of Theorem 1.2. Proofs of (i) and (iii) have been given by Kogan [10], Petersson [14] and Berkovich and Yeşilyurt [4]. Formula (ii) is due to Berkovich and Yeşilyurt [4]. We thank them for providing us with a statement of their theorem. We also thank an unknown referee for her/his suggestions for improving the first version of this paper.

2. Proof of Theorem 1.1. Following [1, p. 178] we set

$$p = p(q) := \frac{\varphi^2(q)-\varphi^2(q^3)}{2\varphi^2(q^3)} \quad \Big(\text{so that } 1+2p = \frac{\varphi^2(q)}{\varphi^2(q^3)}\Big) \tag{2.1}$$

and

$$k = k(q) := \frac{\varphi^3(q^3)}{\varphi(q)}. \tag{2.2}$$

Ramanujan's discriminant function $\Delta(q)$ [15] is defined by

$$\Delta(q) := q \prod_{n=1}^{\infty} (1-q^n)^{24}. \tag{2.3}$$

In [2, p. 36] each of $\Delta(q^r)$ $(r = 1, 2, 3, 4, 6, 12)$ was expressed in terms of p and k, namely

$$\Delta(q) = \frac{1}{16} p(1-p)^{12}(1+p)^4(1+2p)^3(2+p)^3 k^{12}, \tag{2.4}$$

$$\Delta(q^2) = \frac{1}{256} p^2(1-p)^6(1+p)^2(1+2p)^6(2+p)^6 k^{12}, \tag{2.5}$$

$$\Delta(q^3) = \frac{1}{16} p^3(1-p)^4(1+p)^{12}(1+2p)(2+p) k^{12}, \tag{2.6}$$

$$\Delta(q^4) = \frac{1}{65536} p^4(1-p)^3(1+p)(1+2p)^3(2+p)^{12} k^{12}, \tag{2.7}$$

$$\Delta(q^6) = \frac{1}{256} p^6(1-p)^2(1+p)^6(1+2p)^2(2+p)^2 k^{12}, \tag{2.8}$$

$$\Delta(q^{12}) = \frac{1}{65536} p^{12}(1-p)(1+p)^3(1+2p)(2+p)^4 k^{12}. \tag{2.9}$$

From (2.3)-(2.9) we can solve for $\prod_{n=1}^{\infty}(1-q^{rn})$ $(r = 1, 2, 3, 4, 6, 12)$ to obtain the following values.

$$\prod_{n=1}^{\infty}(1-q^n) = q^{-1/24} 2^{-1/6} p^{1/24}(1-p)^{1/2}(1+p)^{1/6}(1+2p)^{1/8}(2+p)^{1/8} k^{1/2}. \tag{2.10}$$

$$\prod_{n=1}^{\infty}(1-q^{2n}) = q^{-1/12} 2^{-1/3} p^{1/12}(1-p)^{1/4}(1+p)^{1/12}(1+2p)^{1/4}(2+p)^{1/4} k^{1/2}. \tag{2.11}$$

$$\prod_{n=1}^{\infty}(1-q^{3n}) = q^{-1/8} 2^{-1/6} p^{1/8}(1-p)^{1/6}(1+p)^{1/2}(1+2p)^{1/24}(2+p)^{1/24} k^{1/2}. \tag{2.12}$$

$$\prod_{n=1}^{\infty}(1-q^{4n}) = q^{-1/6} 2^{-2/3} p^{1/6}(1-p)^{1/8}(1+p)^{1/24}(1+2p)^{1/8}(2+p)^{1/2} k^{1/2}. \tag{2.13}$$

$$\prod_{n=1}^{\infty}(1-q^{6n}) = q^{-1/4} 2^{-1/3} p^{1/4}(1-p)^{1/12}(1+p)^{1/4}(1+2p)^{1/12}(2+p)^{1/12} k^{1/2}. \tag{2.14}$$

$$\prod_{n=1}^{\infty}(1-q^{12n}) = q^{-1/2} 2^{-2/3} p^{1/2}(1-p)^{1/24}(1+p)^{1/8}(1+2p)^{1/24}(2+p)^{1/6} k^{1/2}. \tag{2.15}$$

It is shown in [6, Prop. 2.2, p. 37] that

$$b(q) = \prod_{n=1}^{\infty} \frac{(1-q^n)^3}{1-q^{3n}} \tag{2.16}$$

and

$$c(q) = 3q^{1/3} \prod_{n=1}^{\infty} \frac{(1-q^{3n})^3}{1-q^n}. \tag{2.17}$$

Next for $n \in \mathbb{N}$ we define

$$G_3(n) = \sum_{\substack{d \in \mathbb{N} \\ d \mid n}} \left(\frac{-3}{n/d}\right) d^2, \ H_3(n) = \sum_{\substack{d \in \mathbb{N} \\ d \mid n}} \left(\frac{-3}{d}\right) d^2, \tag{2.18}$$

where $\left(\frac{-3}{k}\right)$ $(k \in \mathbb{N})$ is the Legendre-Jacobi-Kronecker symbol for discriminant -3, that is

$$\left(\frac{-3}{k}\right) = \begin{cases} 1, & \text{if } k \equiv 1 \pmod 3, \\ -1, & \text{if } k \equiv 2 \pmod 3, \\ 0, & \text{if } k \equiv 0 \pmod 3. \end{cases} \tag{2.19}$$

In [3, Theorems 2.2 and 2.3] we used a formula of Carlitz [7, eq. (1.3), p. 168] to prove the following two formulae

$$\sum_{n=1}^{\infty} G_3(n) q^n = q \prod_{n=1}^{\infty} \frac{(1-q^{3n})^9}{(1-q^n)^3} \tag{2.20}$$

and

$$1 - 9\sum_{n=1}^{\infty} H_3(n) q^n = \prod_{n=1}^{\infty} \frac{(1-q^n)^9}{(1-q^{3n})^3}. \tag{2.21}$$

We remark that Dobbie [8] has given an elementary proof of Carlitz's formula. Thus

$$\sum_{n=1}^{\infty} G_3(n) q^n = \frac{1}{27} c^3(q) \tag{2.22}$$

and

$$1 - 9\sum_{n=1}^{\infty} H_3(n) q^n = b^3(q). \tag{2.23}$$

From (2.10), (2.12) and (2.20), we deduce

$$\sum_{n=1}^{\infty} G_3(n) q^n = \frac{1}{2} p(1+p)^4 k^3 \tag{2.24}$$

and from (2.10), (2.12) and (2.21)

$$1-9\sum_{n=1}^{\infty} H_3(n)q^n = \frac{1}{2}(1-p)^4(1+2p)(2+p)k^3. \tag{2.25}$$

From (2.23) and (2.25) we deduce

$$b(q) = 2^{-1/3}(1-p)^{4/3}(1+2p)^{1/3}(2+p)^{1/3}k \tag{2.26}$$

and from (2.22) and (2.24)

$$c(q) = 2^{-1/3}3p^{1/3}(1+p)^{4/3}k. \tag{2.27}$$

From (2.16) we obtain

$$\begin{aligned}
b(-q) &= \prod_{n=1}^{\infty} \frac{(1-(-1)^n q^n)^3}{(1-(-1)^n q^{3n})} \\
&= \prod_{n=1}^{\infty} \frac{(1-q^{2n})^3(1+q^{2n-1})^3}{(1-q^{6n})(1+q^{6n-3})} \\
&= \prod_{n=1}^{\infty} \frac{(1-q^{2n})^3}{(1-q^{6n})}\frac{(1+q^n)^3}{(1+q^{2n})^3}\frac{(1+q^{6n})}{(1+q^{3n})} \\
&= \prod_{n=1}^{\infty} \frac{(1-q^{2n})^3}{(1-q^{6n})}\frac{(1-q^{2n})^3}{(1-q^n)^3}\frac{(1-q^{2n})^3}{(1-q^{4n})^3}\frac{(1-q^{12n})}{(1-q^{6n})}\frac{(1-q^{3n})}{(1-q^{6n})} \\
&= \prod_{n=1}^{\infty} \frac{(1-q^{2n})^9(1-q^{3n})(1-q^{12n})}{(1-q^n)^3(1-q^{4n})^3(1-q^{6n})^3}.
\end{aligned}$$

Appealing to (2.10)-(2.15), we deduce

$$b(-q) = 2^{-1/3}(1-p)^{1/3}(1+2p)^{4/3}(2+p)^{1/3}k. \tag{2.28}$$

Similarly we deduce

$$c(-q) = -2^{-1/3}3p^{1/3}(1+p)^{1/3}k, \tag{2.29}$$

$$b(q^2) = 2^{-2/3}(1-p)^{2/3}(1+2p)^{2/3}(2+p)^{2/3}k, \tag{2.30}$$

$$c(q^2) = 2^{-2/3}3p^{2/3}(1+p)^{2/3}k, \tag{2.31}$$

$$b(q^4) = 2^{-4/3}(1-p)^{1/3}(1+2p)^{1/3}(2+p)^{4/3}k, \tag{2.32}$$

$$c(q^4) = 2^{-4/3}3p^{4/3}(1+p)^{1/3}k. \tag{2.33}$$

The values in (2.26)-(2.33) are in agreement with those in [1, pp. 178-179]. (In Theorem 8 of [1] $2^{1/3}$ should be replaced by $2^{-1/3}$.) From (2.28)-(2.31), (2.1) and (2.2), we have

$$\frac{1}{9}b^3(-q)+\frac{8}{9}b^3(q^2)-\frac{1}{3}c^3(-q)+\frac{8}{3}c^3(q^2)$$

$$
\begin{aligned}
&= \frac{1}{18}(1-p)(1+2p)^4(2+p)k^3 + \frac{2}{9}(1-p)^2(1+2p)^2(2+p)^2k^3 \\
&\qquad\qquad\qquad\qquad + \frac{9}{2}p(1+p)k^3 + 18p^2(1+p)^2k^3 \\
&= (1+2p)^4k^3 \\
&= \left(\frac{\varphi^2(q)}{\varphi^2(q^3)}\right)^4 \left(\frac{\varphi^3(q^3)}{\varphi(q)}\right)^3 \\
&= \varphi^5(q)\varphi(q^3),
\end{aligned}
$$

which is identity (i) of Theorem 1.1. Identity (iii) follows in a similar manner using the identity

$$
\begin{aligned}
\frac{1}{18}&(1-p)(1+2p)^4(2+p) + \frac{2}{9}(1-p)^2(1+2p)^2(2+p)^2 \\
&\qquad + \frac{1}{2}p(1+p) + 2p^2(1+p)^2 \\
&\qquad = (1+2p)^2.
\end{aligned}
$$

Finally we prove identity (ii). By (2.26), (2.32), (2.27), (2.33), (2.11), (2.14), (2.1) and (2.2), we have

$$
\begin{aligned}
\frac{1}{9}&b^3(q) + \frac{8}{9}b^3(q^4) + \frac{1}{9}c^3(q) + \frac{8}{9}c^3(q^4) + 4q\prod_{n=1}^{\infty}(1-q^{2n})^3(1-q^{6n})^3 \\
&= \frac{1}{18}(1-p)^4(1+2p)(2+p)k^3 + \frac{1}{18}(1-p)(1+2p)(2+p)^4k^3 \\
&\quad + \frac{3}{2}p(1+p)^4k^3 + \frac{3}{2}p^4(1+p)k^3 + p(1-p)(1+p)(1+2p)(2+p)k^3 \\
&= (1+2p)^3k^3 \\
&= \left(\frac{\varphi^2(q)}{\varphi^2(q^3)}\right)^3 \left(\frac{\varphi^3(q^3)}{\varphi(q)}\right)^3 \\
&= \varphi^3(q)\varphi^3(q^3),
\end{aligned}
$$

as required. ∎

3. Proof of Theorem 1.2. We just prove (ii) as (i) and (iii) can be proved similarly.

Recall that the integers $k(n)$ $(n \in \mathbb{N})$ are defined in (1.10). By (1.8), (1.9), Theorem 1.1(ii), (2.23), (2.22) and (1.10), we have

$$
\begin{aligned}
1 + &\sum_{n=1}^{\infty} N(1,1,1,3,3,3;n)q^n \\
&= \sum_{n=0}^{\infty} N(1,1,1,3,3,3;n)q^n \\
&= \varphi^3(q)\varphi^3(q^3)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{9}b^3(q) + \frac{8}{9}b^3(q^4) + \frac{1}{9}c^3(q) + \frac{8}{9}c^3(q^4) + 4q\prod_{n=1}^{\infty}(1-q^{2n})^3(1-q^{6n})^3 \\
&= \frac{1}{9}\big(1 - 9\sum_{n=1}^{\infty} H_3(n)q^n\big) + \frac{8}{9}\big(1 - 9\sum_{n=1}^{\infty} H_3(n)q^{4n}\big) \\
&+3\sum_{n=1}^{\infty} G_3(n)q^n + 24\sum_{n=1}^{\infty} G_3(n)q^{4n} + 4\sum_{n=1}^{\infty} k(n)q^n \\
&= 1 + \sum_{n=1}^{\infty}(3G_3(n) + 24G_3(n/4) - H_3(n) - 8H_3(n/4) + 4k(n))q^n.
\end{aligned}
$$

Equating coefficients of q^n $(n \in \mathbb{N})$, we obtain

$$
N(1,1,1,3,3,3;n) = 3G_3(n) + 24G_3(n/4) - H_3(n) - 8H_3(n/4) + 4k(n).
$$

Writing $n = 2^\alpha 3^\beta N$, where $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $\gcd(N,6) = 1$, it is easy to show that

$$
\begin{aligned}
G_3(n) &= \frac{1}{5}(2^{2\alpha+2} + (-1)^\alpha)3^{2\beta}G_3(N), \\
H_3(n) &= \frac{1}{5}(-1)^\alpha(2^{2\alpha+2} + (-1)^\alpha)H_3(N), \\
G_3(N) &= \Big(\frac{-3}{N}\Big)H_3(N).
\end{aligned}
$$

Set

$$
f(n) = \begin{cases} 0, & \text{if } n \equiv 1 \pmod 2 \text{ or } n \equiv 2 \pmod 4, \\ 1, & \text{if } n \equiv 0 \pmod 4. \end{cases}
$$

Then

$$
\begin{aligned}
&3G_3(n) + 24G_3(n/4) - H_3(n) - 8H_3(n/4) \\
&\quad = \Big(\frac{1}{5}(2^{2\alpha+2} + (-1)^\alpha)3^{2\beta+1} + \frac{1}{5}f(n)(2^{2\alpha+1} + (-1)^\alpha 8)3^{2\beta+1} \\
&\qquad\qquad - \frac{1}{5}(-1)^\alpha(2^{2\alpha+2} + (-1)^\alpha)\Big(\frac{-3}{N}\Big) \\
&\qquad\qquad\qquad - \frac{1}{5}f(n)(-1)^\alpha(2^{2\alpha+1} + (-1)^\alpha 8)\Big(\frac{-3}{N}\Big)\Big)G_3(N) \\
&\quad = \frac{1}{5}\Big((2^{2\alpha+2} + (-1)^\alpha) + f(n)(2^{2\alpha+1} + (-1)^\alpha 8)\Big) \\
&\qquad\qquad \times\Big(3^{2\beta+1} - (-1)^\alpha\Big(\frac{-3}{N}\Big)\Big)G_3(N).
\end{aligned}
$$

Now

$$
\begin{aligned}
&(2^{2\alpha+2} + (-1)^\alpha) + f(n)(2^{2\alpha+1} + (-1)^\alpha 8) \\
&\quad = \begin{cases} 5, & \text{if } \alpha = 0, \\ 15, & \text{if } \alpha = 1, \\ 3\cdot 2^{2\alpha+1} + 9(-1)^\alpha, & \text{if } \alpha \ge 2, \end{cases}
\end{aligned}
$$

$$= \begin{cases} 5, & \text{if } \alpha = 0, \\ 3 \cdot 2^{2\alpha+1} + 9(-1)^{\alpha}, & \text{if } \alpha \geq 1, \end{cases}$$

$$= \frac{1}{2}(3 \cdot 2^{2\alpha+1} + 9(-1)^{\alpha})(1 + (-1)^{2^{\alpha}}) + \frac{5}{2}(1 - (-1)^{2^{\alpha}})$$

and

$$\begin{aligned} G_3(N) &= \sum_{\substack{d \in \mathbb{N} \\ d \mid N}} \left(\frac{-3}{N/d}\right) d^2 \\ &= N^2 \sum_{\substack{d \in \mathbb{N} \\ d \mid N}} \left(\frac{-3}{d}\right) \frac{1}{d^2} \\ &= N^2 \prod_{p \mid N} \left(\sum_{r=0}^{\alpha_p} \left(\frac{-3}{p}\right)^r p^{-2r} \right) \\ &= N^2 \prod_{p \mid N} \frac{1 - \left(\frac{-3}{p}\right)^{\alpha_p + 1} p^{-2\alpha_p - 2}}{1 - \left(\frac{-3}{p}\right) p^{-2}}. \end{aligned}$$

Putting these results together, we obtain part (ii) of Theorem 1.2. ∎

4. Concluding remark. We show that the identity (1.4) of Borwein, Borwein and Garvan in conjuction with (2.22) and (2.23) gives an explicit formula for

$$\begin{aligned} N(n) := \operatorname{card}\{&(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{Z}^6 \mid \\ &n = x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2 + x_5^2 + x_5x_6 + x_6^2\} \end{aligned} \tag{4.1}$$

Appealing to (4.1), (1.1), (1.4), (2.23) and (2.22), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} N(n) q^n &= a^3(q) \\ &= b^3(q) + c^3(q) \\ &= 1 - 9 \sum_{n=1}^{\infty} H_3(n) q^n + 27 \sum_{n=1}^{\infty} G_3(n) q^n \\ &= 1 + \sum_{n=1}^{\infty} (27G_3(n) - 9H_3(n)) q^n. \end{aligned}$$

Equating coefficients of q^n $(n \in \mathbb{N})$ we obtain

$$N(n) = 27G_3(n) - 9H_3(n), \ n \in \mathbb{N}. \tag{4.2}$$

The formula (4.2) is given in Lomadze [13, p. 12] with a reference to Petersson [14].

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