JP Journal of Algebra, Number Theory and Applications
Volume 9, Number 1, 2007, Pages 37-53
This paper is available online at http://www.pphmj.com

ON THE QUATERNARY FORMS $x^2+y^2+z^2+5t^2$, $x^2+y^2+5z^2+5t^2$ AND $x^2+5y^2+5z^2+5t^2$

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Abstract

Simple proofs are given of the formulae for the number of representations of a positive integer by each of the three quaternary quadratic forms $x^2+y^2+z^2+5t^2$, $x^2+y^2+5z^2+5t^2$ and $x^2+5y^2+5z^2+5t^2$.

1. Introduction

Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{Z}$ denote the set of integers. Let $n \in \mathbb{N}$. Set $n = 2^{\alpha}5^{\beta}N$, where $\alpha \in \mathbb{N} \cup \{0\}$, $\beta \in \mathbb{N} \cup \{0\}$, $N \in \mathbb{N}$ and $\gcd(N, 10) = 1$. In 1864 Liouville [11, 12] stated without proof formulae equivalent to

2000 Mathematics Subject Classification: 11E25.

Keywords and phrases: Liouville's formulae, Kloosterman's formulae, quaternary quadratic forms, representations.

The third author was supported by research grant A-7233 from the Natural Sciences and Engineering Research Council of Canada.

Received August 22, 2007

$$N(1, 1, 1, 5; n) := \text{card}\{(x, y, z, t) \in \mathbb{Z}^4 \mid n = x^2 + y^2 + z^2 + 5t^2\}$$

$$= \sum_{d \mid n} (-1)^{n+d} \left(\frac{5}{d}\right) d + 5 \sum_{d \mid n} (-1)^{n+d} \left(\frac{5}{n/d}\right) d \tag{1.1}$$

and

$$N(1, 5, 5, 5; n) := \text{card}\{(x, y, z, t) \in \mathbb{Z}^4 \mid n = x^2 + 5y^2 + 5z^2 + 5t^2\}$$

$$= \sum_{d \mid n} (-1)^{n+d} \left(\frac{5}{d}\right) d + \sum_{d \mid n} (-1)^{n+d} \left(\frac{5}{n/d}\right) d, \tag{1.2}$$

where $\left(\frac{5}{k}\right)$ $(k \in \mathbb{N})$ is the Legendre-Jacobi-Kronecker symbol for discriminant 5, that is

$$\left(\frac{5}{k}\right) = \begin{cases} 0, & \text{if } k \equiv 0 \pmod 5, \\ 1, & \text{if } k \equiv 1, 4 \pmod 5, \\ -1, & \text{if } k \equiv 2, 3 \pmod 5. \end{cases}$$

Trivially $N(1, 1, 1, 5; 0) = N(1, 5, 5, 5; 0) = 1$. A year later Liouville [13] gave the formula

$$N(1, 1, 5, 5; n) := \text{card}\{(x, y, z, t) \in \mathbb{Z}^4 \mid n = x^2 + y^2 + 5z^2 + 5t^2\}$$

$$= 2(5^{\beta+1} - 3)\sigma(N), \quad \text{if } \alpha \geq 1,$$

but did not give a result when $\alpha = 0$. Again no proof was given. According to Dickson [6, Vol. 3, p. 232], Petr [20, p. 20] evaluated $N(1, 1, 1, 5; 8n)$ $(n \in \mathbb{N})$ in terms of the class-number of binary quadratic forms. Also according to Dickson [6, Vol. 3, p. 232], Petr [21] enumerated by the use of theta functions the solutions of $x^2 + y^2 + z^2 + 5t^2 = n$. In 1926 Kloosterman [10, p. 173] gave the following formula for $N(1, 1, 5, 5; n)$ $(n \in \mathbb{N})$, namely

$$N(1, 1, 5, 5; n) = \begin{cases} 2(5^{\beta+1} - 3)\sigma(N), & \text{if } \alpha \geq 1, \\ \frac{2}{3}(5^{\beta+1} - 3)\sigma(N) + \frac{8}{3}c(n), & \text{if } \alpha = 0, \end{cases} \tag{1.3}$$

where $c(n)$ $(n \in \mathbb{N})$ is given by

$$q\prod_{n=1}^{\infty}(1-q^{2n})^2(1-q^{10n})^2 = \sum_{n=1}^{\infty}c(n)q^n, \tag{1.4}$$

and

$$\sigma(n) = \sum_{\substack{d\in\mathbb{N} \\ d\mid n}} d, \quad n \in \mathbb{N}.$$

(If $n \notin \mathbb{N}$, then we set $\sigma(n) = 0$.) Trivially $N(1, 1, 5, 5; 0) = 1$. Kloosterman did not give his proof of (1.3) and remarked that the proof is *very complicated* [10, p. 173]. The first published proof of (1.3) was given by Lomadse [14, Satz 1(a), p. 152] and the first proof of (1.1) by Demuth [5, pp. 245-247]. Proofs of (1.1)-(1.3) were given in Petersson's book [19, Satz 11.1, p. 107; Satz 11.2, p. 108]. Proofs can also be given using Ramanujan's modular equations of degree 5 as given by Berndt in [3, Part III, Chapter 9; Part V, Chapter 36]. Our purpose in this paper is to give simple proofs of (1.1)-(1.3) based upon an identity of Bailey [1] and some modular equations of degree 5 given by Ramanujan [22], which were proved by Berndt [3, Part III]. Berkovich [2] of the University of Florida has informed the third author that he and Hamza Yesilyurt have also given proofs of (1.1) and (1.2).

2. Notation

Let q denote a complex number with $|q| < 1$. For $k \in \mathbb{N}$ we define

$$E_k = E_k(q) := \prod_{n=1}^{\infty}(1-q^{kn}). \tag{2.1}$$

The theta functions $\varphi(q)$ and $\psi(q)$ are defined by

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} \tag{2.2}$$

and

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}. \tag{2.3}$$

The infinite product representations of $\varphi(q)$ and $\psi(q)$ are due to Jacobi [9], namely

$$\varphi(q) = \frac{E_2^5}{E_1^2 E_4^2} \tag{2.4}$$

and

$$\psi(q) = \frac{E_2^2}{E_1}, \tag{2.5}$$

see for example [8, p. 284]. As

$$E_k(-q) = \begin{cases} E_k(q), & \text{if } k \text{ is even,} \\ \dfrac{E_{2k}^3(q)}{E_k(q)E_{4k}(q)}, & \text{if } k \text{ is odd,} \end{cases} \tag{2.6}$$

we deduce from (2.4)-(2.6)

$$\psi(-q) = \frac{E_1 E_4}{E_2} \tag{2.7}$$

and

$$\varphi(-q) = \frac{E_1^2}{E_2}. \tag{2.8}$$

In the notation of Berndt [3, Part III, pp. 36, 37] we have

$$\chi(-q) = \frac{E_1}{E_2}, \quad f(-q) = E_1. \tag{2.9}$$

Thus, by (2.6), we have

$$\chi(q) = \frac{E_2^2}{E_1 E_4}, \quad f(q) = \frac{E_2^3}{E_1 E_4}. \tag{2.10}$$

3. Ramanujan's Modular Equations of Degree 5

Ramanujan [22, Entry 19, p. 295] asserted that (with q replaced by $-q$)

$$\psi^2(q) - 5q\psi^2(q^5) = \frac{\varphi^2(-q)}{\chi(-q)\chi(-q^5)}. \tag{3.1}$$

A proof has been given by Berndt [3, Part V, p. 366]. Appealing to (2.4)-(2.7) and (3.1), we obtain

$$\frac{E_2^4}{E_1^2} - 5q\,\frac{E_{10}^4}{E_5^2} = \frac{E_1^3 E_{10}}{E_2 E_5}. \tag{3.2}$$

Multiplying both sides of (3.2) by $E_1^2 E_2 E_5^2$, we have

Theorem 3.1.

$$E_2^5 E_5^2 - 5q E_1^2 E_2 E_{10}^4 = E_1^5 E_5 E_{10}.$$

Ramanujan [22, Entry 18, p. 295] also asserted (with q replaced by $-q$)

$$\psi^2(q) - q\psi^2(q^5) = \frac{f(-q^5)\varphi(-q^5)}{\chi(-q)}. \tag{3.3}$$

A proof is given in [3, Part V, pp. 365-366]. Appealing to (2.5)-(2.7) and (3.3), we obtain

$$\frac{E_2^4}{E_1^2} - q\,\frac{E_{10}^4}{E_5^2} = \frac{E_2 E_5^3}{E_1 E_{10}}. \tag{3.4}$$

Multiplying both sides of (3.4) by $E_1^2 E_5^2 E_{10}$, we have

Theorem 3.2.

$$E_2^4 E_5^2 E_{10} - q E_1^2 E_{10}^5 = E_1 E_2 E_5^5.$$

Our next theorem is also due to Ramanujan [22].

Theorem 3.3.

$$\varphi^2(q) - \varphi^2(q^5) = 4q\,\frac{E_2^2 E_5 E_{20}}{E_1 E_4}.$$

Proof. Ramanujan asserted and Berndt [3, Part III, p. 258] proved that

$$\varphi^2(q) - \varphi^2(q^5) = 4q\chi(q) f(-q^5) f(-q^{20}).$$

By (2.9) and (2.10) we obtain

$$\chi(q) f(-q^5) f(-q^{20}) = \frac{E_2^2}{E_1 E_4} E_5 E_{20}$$

and the asserted result follows.

Theorem 3.4.

$$\varphi^2(q) - 5\varphi^2(q^5) = -4 \frac{E_1 E_4 E_{10}^2}{E_5 E_{20}}.$$

Proof. From [3, Part III, p. 259] we have

$$5 \frac{\varphi^2(q^5)}{\varphi^2(q)} - 1 = 4 \frac{\chi(q^5)}{\chi^5(q)}.$$

Thus

$$\varphi^2(q) - 5\varphi^2(q^5) = -4 \frac{\chi(q^5)}{\chi^5(q)} \varphi^2(q).$$

By (2.4) and (2.10) we deduce

$$\frac{\chi(q^5)}{\chi^5(q)} \varphi^2(q) = \frac{E_1 E_4 E_{10}^2}{E_5 E_{20}},$$

and the asserted result follows.

4. Bailey's Identity

The identity of Theorem 4.1 is implicit in the work of Bailey [1, eqs. (4) and (5)], who obtained it from a formula for the difference of two values of the Weierstrass $\wp$-function. Dobbie [7] has given an elementary proof of Bailey's identity.

Theorem 4.1. *Let a and b be complex numbers such that $a \neq 0$, $b \neq 0$, $a \neq b$, $ab \neq 1$, $a \neq q^n$ for any integer n and $b \neq q^n$ for any integer n.*

Then

$$\prod_{n=1}^{\infty} \frac{(1-abq^n)(1-a^{-1}b^{-1}q^n)(1-ab^{-1}q^n)(1-a^{-1}bq^n)(1-q^n)^4}{(1-aq^n)^2(1-a^{-1}q^n)^2(1-bq^n)^2(1-b^{-1}q^n)^2}$$

$$= 1 + \frac{(1-a)^2(1-b)^2}{(a-b)(1-ab)} \sum_{n=1}^{\infty} \left(\sum_{d \mid n} (a^d + a^{-d} - b^d - b^{-d}) d \right) q^n.$$

Let $\omega = e^{2\pi i/5}$ so that

$$\omega^5 = 1, \quad \omega + \omega^2 + \omega^3 + \omega^4 = -1 \tag{4.1}$$

and

$$\omega^d - \omega^{2d} - \omega^{3d} + \omega^{4d} = \left(\frac{5}{d}\right)\sqrt{5}, \quad d \in \mathbb{N}. \tag{4.2}$$

We choose $a = -\omega$ and $b = -\omega^2$ in Bailey's identity. As

$$\begin{aligned} a^d + a^{-d} - b^d - b^{-d} &= (-1)^d(\omega^d - \omega^{2d} - \omega^{3d} + \omega^{4d}) \\ &= (-1)^d \left(\frac{5}{d}\right)\sqrt{5}, \quad d \in \mathbb{N}, \end{aligned}$$

$$\frac{(1-a)^2(1-b)^2}{(a-b)(1-ab)} = \frac{1}{\sqrt{5}},$$

and

$$\prod_{n=1}^{\infty} \frac{(1-abq^n)(1-a^{-1}b^{-1}q^n)(1-ab^{-1}q^n)(1-a^{-1}bq^n)(1-q^n)^4}{(1-aq^n)^2(1-a^{-1}q^n)^2(1-bq^n)^2(1-b^{-1}q^n)^2}$$

$$= \prod_{n=1}^{\infty} \frac{(1-\omega q^n)(1-\omega^2 q^n)(1-\omega^3 q^n)(1-\omega^4 q^n)(1-q^n)^4}{(1+\omega q^n)^2(1+\omega^2 q^n)^2(1+\omega^3 q^n)^2(1+\omega^4 q^n)^2}$$

$$= \prod_{n=1}^{\infty} \frac{(1-q^{5n})(1-q^n)^3}{\left(\dfrac{1+q^{5n}}{1+q^n}\right)^2}$$

$$= \prod_{n=1}^{\infty} \frac{(1-q^{5n})(1-q^n)^3(1-q^{2n})^2(1-q^{5n})^2}{(1-q^{10n})^2(1-q^n)^2}$$

$$= \prod_{n=1}^{\infty} \frac{(1-q^n)(1-q^{2n})^2(1-q^{5n})^3}{(1-q^{10n})^2},$$

we obtain

Theorem 4.2.

$$\frac{E_1E_2^2E_5^3}{E_{10}^2} = 1 + \sum_{n=1}^{\infty}\left(\sum_{d \mid n}(-1)^d\left(\frac{5}{d}\right)d\right)q^n.$$

Next we replace q by q^5 in Bailey's identity, and then take $a = -q$ and $b = -q^3$. We have

$$\prod_{n=1}^{\infty} \frac{(1-abq^{5n})(1-a^{-1}b^{-1}q^{5n})(1-ab^{-1}q^{5n})(1-a^{-1}bq^{5n})(1-q^{5n})^4}{(1-aq^{5n})^2(1-a^{-1}q^{5n})^2(1-bq^{5n})^2(1-b^{-1}q^{5n})^2}$$

$$= \prod_{n=1}^{\infty} \frac{(1-q^{5n+4})(1-q^{5n-4})(1-q^{5n-2})(1-q^{5n+2})(1-q^{5n})^4}{(1+q^{5n+1})^2(1+q^{5n-1})^2(1+q^{5n+3})^2(1+q^{5n-3})^2}$$

$$= \frac{(1+q)^2(1+q^3)^2}{(1-q^2)(1-q^4)} \prod_{n=1}^{\infty} \frac{(1-q^{5n-4})(1-q^{5n-3})(1-q^{5n-2})(1-q^{5n-1})(1-q^{5n})^4}{(1+q^{5n-4})^2(1+q^{5n-3})^2(1+q^{5n-2})^2(1+q^{5n-1})^2}$$

$$= \frac{(1+q)^2(1+q^3)^2}{(1-q^2)(1-q^4)} \prod_{n=1}^{\infty} \frac{(1-q^n)(1-q^{5n})^3}{\left(\dfrac{1+q^n}{1+q^{5n}}\right)^2}$$

$$= \frac{(1+q)^2(1+q^3)^2}{(1-q^2)(1-q^4)} \prod_{n=1}^{\infty} \frac{(1-q^n)^3(1-q^{5n})(1-q^{10n})^2}{(1-q^{2n})^2}.$$

Also

$$\frac{(1-a)^2(1-b)^2}{(a-b)(1-ab)} = -\frac{(1+q)^2(1+q^3)^2}{(1-q^2)(1-q^4)}.$$

Then, by Theorem 4.1, we obtain

$$q\,\frac{E_1^3E_5E_{10}^2}{E_2^2} = q\,\frac{(1-q^2)(1-q^4)}{(1+q^2)(1+q^3)^2} - \sum_{n=1}^{\infty}\left(\sum_{d\mid n}(-1)^d(q^d+q^{-d}-q^{3d}-q^{-3d})d\right)q^{5n}.$$

Now

$$\begin{aligned}
\sum_{n=1}^{\infty}\left(\sum_{d\mid n}(-1)^d q^d d\right)q^{5n} &= \sum_{n=1}^{\infty}\sum_{d\mid n}(-1)^d dq^{d+5n}\\
&= \sum_{d,e=1}^{\infty}(-1)^d dq^{d+5de}\\
&= \sum_{d,e=1}^{\infty}(-1)^d dq^{(5e+1)d}\\
&= \sum_{d=1}^{\infty}\sum_{\substack{f=6\\ f\equiv 1 \pmod 5}}^{\infty}(-1)^d dq^{fd}\\
&= \sum_{d=1}^{\infty}\sum_{\substack{f=1\\ f\equiv 1 \pmod 5}}^{\infty}(-1)^d dq^{fd} - \sum_{d=1}^{\infty}(-1)^d dq^d,
\end{aligned}$$

that is

$$\sum_{n=1}^{\infty}\left(\sum_{d\mid n}(-1)^d q^d d\right)q^{5n} = \sum_{n=1}^{\infty}\left(\sum_{\substack{d\mid n\\ n/d\equiv 1 \pmod 5}}(-1)^d d\right)q^n + \frac{q}{(1+q)^2}.$$

Similarly

$$\sum_{n=1}^{\infty}\left(\sum_{d\mid n}(-1)^d q^{-d} d\right)q^{5n} = \sum_{n=1}^{\infty}\left(\sum_{\substack{d\mid n\\ n/d\equiv 4 \pmod 5}}(-1)^d d\right)q^n,$$

$$\sum_{n=1}^{\infty}\left(\sum_{d\mid n}(-1)^d q^{3d} d\right)q^{5n} = \sum_{n=1}^{\infty}\left(\sum_{\substack{d\mid n\\ n/d\equiv 3 \pmod 5}}(-1)^d d\right)q^n + \frac{q^3}{(1+q^3)^2},$$

$$\sum_{n=1}^{\infty}\left(\sum_{d\mid n}(-1)^d q^{-3d} d\right)q^{5n} = \sum_{n=1}^{\infty}\left(\sum_{\substack{d\mid n\\ n/d\equiv 2 \pmod 5}}(-1)^d d\right)q^n.$$

Thus

$$\sum_{n=1}^{\infty}\left(\sum_{d\mid n}(-1)^{d}(q^{d}+q^{-d}-q^{3d}-q^{-3d})d\right)q^{5n}$$

$$=\sum_{n=1}^{\infty}\left(\sum_{d\mid n}(-1)^{d}\left(\frac{5}{n/d}\right)d\right)q^{n}+\frac{q}{(1+q)^{2}}-\frac{q^{3}}{(1+q^{3})^{2}}$$

$$=\sum_{n=1}^{\infty}\left(\sum_{d\mid n}(-1)^{d}\left(\frac{5}{n/d}\right)d\right)q^{n}+\frac{q(1-q^{2})(1-q^{4})}{(1+q)^{2}(1+q^{3})^{2}}.$$

We have proved the following identity:

Theorem 4.3.

$$q\,\frac{E_1^3E_5E_{10}^2}{E_2^2}=-\sum_{n=1}^{\infty}\left(\sum_{d\mid n}(-1)^{d}\left(\frac{5}{n/d}\right)d\right)q^{n}.$$

5. Evaluation of $N(1, 1, 1, 5;\ n)$

We use Theorems 3.1, 4.2 and 4.3 to determine $N(1,1,1,5;n)$ $(n\in\mathbb{N})$.

Theorem 5.1. *Let* $n\in\mathbb{N}$. *Then*

$$N(1,1,1,5;n)=\sum_{d\mid n}(-1)^{n+d}\left(\frac{5}{d}\right)d+5\sum_{d\mid n}(-1)^{n+d}\left(\frac{5}{n/d}\right)d.$$

Proof. We have

$$\sum_{n=0}^{\infty}N(1,1,1,5;n)(-q)^{n}$$

$$=\varphi^{3}(-q)\varphi(-q^{5}) \qquad \text{(by (2.2))}$$

$$=\frac{E_1^6E_5^2}{E_2^3E_{10}} \qquad \text{(by (2.8))}$$

$$= \frac{E_1E_5}{E_2^3E_{10}^2} \cdot E_1^5E_5E_{10}$$

$$= \frac{E_1E_5}{E_2^3E_{10}^2}(E_2^5E_5^2 - 5qE_1^2E_2E_{10}^4) \qquad \text{(by Theorem 3.1)}$$

$$= \frac{E_1E_2^2E_5^3}{E_{10}^2} - 5q\frac{E_1^3E_5E_{10}^2}{E_2^2}$$

$$= 1 + \sum_{n=1}^{\infty}\left(\sum_{d\mid n}(-1)^d\left(\frac{5}{d}\right)d\right)q^n + 5\sum_{n=1}^{\infty}\left(\sum_{d\mid n}(-1)^d\left(\frac{5}{n/d}\right)d\right)q^n,$$

by Theorems 4.2 and 4.3. Equating coefficients of q^n $(n \in \mathbb{N})$, we obtain

$$(-1)^n N(1, 1, 1, 5; n) = \sum_{d\mid n}(-1)^d\left(\frac{5}{d}\right)d + 5\sum_{d\mid n}(-1)^d\left(\frac{5}{n/d}\right)d$$

from which Theorem 5.1 follows.

6. Evaluation of $N(1, 5, 5, 5; n)$

We use Theorems 3.2, 4.2 and 4.3 to determine $N(1, 5, 5, 5; n)$ $(n \in \mathbb{N})$.

Theorem 6.1. *Let* $n \in \mathbb{N}$. *Then*

$$N(1, 5, 5, 5; n) = \sum_{d\mid n}(-1)^{n+d}\left(\frac{5}{d}\right)d + \sum_{d\mid n}(-1)^{n+d}\left(\frac{5}{n/d}\right)d.$$

Proof. We have

$$\sum_{n=0}^{\infty} N(1, 5, 5, 5; n)(-q)^n$$

$$= \varphi(-q)\varphi^3(-q^5) \qquad \text{(by (2.2))}$$

$$= \frac{E_1^2E_5^6}{E_2E_{10}^3} \qquad \text{(by (2.8))}$$

$$= \frac{E_1 E_5}{E_2^2 E_{10}^3} \cdot E_1 E_2 E_5^5$$

$$= \frac{E_1 E_5}{E_2^2 E_{10}^3} (E_2^4 E_5^2 E_{10} - q E_1^2 E_{10}^5) \qquad \text{(by Theorem 3.2)}$$

$$= \frac{E_1 E_2^2 E_5^3}{E_{10}^2} - q \frac{E_1^3 E_5 E_{10}^2}{E_2^2}$$

$$= 1 + \sum_{n=1}^{\infty} \left(\sum_{d \mid n} (-1)^d \left(\frac{5}{d} \right) d \right) q^n + \sum_{n=1}^{\infty} \left(\sum_{d \mid n} (-1)^d \left(\frac{5}{n/d} \right) d \right) q^n,$$

by Theorems 4.2 and 4.3.

7. Evaluation of $N(1, 1, 5, 5; n)$

We use Theorems 3.3 and 3.4 to determine $N(1, 1, 5, 5; n)$ $(n \in \mathbb{N})$.

Theorem 7.1. *Let* $n \in \mathbb{N}$. *Set* $n = 2^{\alpha} 5^{\beta} N$, *where* $\alpha \in \mathbb{N} \cup \{0\}$, $\beta \in \mathbb{N} \cup \{0\}$, $N \in \mathbb{N}$ *and* $\gcd = (N, 10) = 1$. *Then*

$$N(1, 1, 5, 5; n) = \begin{cases} 2(5^{\beta+1} - 3)\sigma(N), & \text{if } \alpha \geq 1, \\ \frac{2}{3}(5^{\beta+1} - 3)\sigma(N) + \frac{8}{3} c(n), & \text{if } \alpha = 0, \end{cases} \qquad (7.1)$$

where $c(n)$ $(n \in \mathbb{N})$ *is given by* (1.4).

Proof. By Theorems 3.3 and 3.4 we have

$$\varphi^4(q) - 6\varphi^2(q)\varphi^2(q^5) + 5\varphi^4(q^5)$$

$$= (\varphi^2(q) - \varphi^2(q^5))(\varphi^2(q) - 5\varphi^2(q^5))$$

$$= 4q \frac{E_2^2 E_5 E_{20}}{E_1 E_4} \cdot \frac{-4 E_1 E_4 E_{10}^2}{E_5 E_{20}}$$

$$= -16 q E_2^2 E_{10}^2.$$

Thus

$$\sum_{n=0}^{\infty} N(1,\ 1,\ 5,\ 5;\ n)q^n$$

$$= \varphi^2(q)\varphi^2(q^5) \qquad \text{(by (2.2))}$$

$$= \frac{1}{6}\varphi^4(q) + \frac{5}{6}\varphi^4(q^5) + \frac{8}{3}qE_2^2E_{10}^2$$

$$= \frac{1}{6}\sum_{n=0}^{\infty} N(1,1,1,1;\ n)q^n + \frac{5}{6}\sum_{n=0}^{\infty} N(1,1,1,1;\ n)q^{5n} + \frac{8}{3}\sum_{n=1}^{\infty} c(n)q^n, \quad \text{(by (1.4))}$$

where for $n \in \mathbb{N} \cup \{0\}$

$$N(1,\ 1,\ 1,\ 1;\ n) = \text{card}\{(x,\ y,\ z,\ t) \in \mathbb{Z}^4 \mid n = x^2 + y^2 + z^2 + t^2\}.$$

Equating coefficients of q^n $(n \in \mathbb{N})$, we obtain

$$N(1,\ 1,\ 5,\ 5;\ n) = \frac{1}{6}N(1,\ 1,\ 1,\ 1;\ n) + \frac{5}{6}N(1,\ 1,\ 1,\ 1;\ n/5) + \frac{8}{3}c(n).$$

Now it is a classical result of Jacobi [9] (see for example [4, p. 59]) that

$$N(1,\ 1,\ 1,\ 1;\ n) = 8\sigma(n) - 32\sigma(n/4), \quad n \in \mathbb{N}. \tag{7.2}$$

Hence

$$\begin{aligned} N(1,\ 1,\ 5,\ 5;\ n) = {} & \frac{4}{3}\sigma(n) - \frac{16}{3}\sigma(n/4) + \frac{20}{3}\sigma(n/5) \\ & - \frac{80}{3}\sigma(n/20) + \frac{8}{3}c(n), \quad n \in \mathbb{N}. \end{aligned} \tag{7.3}$$

Set $n = 2^\alpha 5^\beta N$, where $\alpha, \beta \in \mathbb{N} \cup \{0\}$, $N \in \mathbb{N}$ and $\gcd(N, 10) = 1$. Suppose first that $\alpha \geq 1$. From (1.4) we see that

$$c(n) = 0 \quad \text{for } n \equiv 0 \pmod 2. \tag{7.4}$$

As

$$\sigma(n) = \sigma(2^\alpha)\sigma(5^\beta)\sigma(N) = \frac{1}{4}(2^{\alpha+1} - 1)(5^{\beta+1} - 1)\sigma(N)$$

we have

$$N(1, 1, 5, 5; n) = \frac{1}{3}(2^{\alpha+1}-1)(5^{\beta+1}-1)\sigma(N) - \frac{4}{3}(2^{\alpha-1}-1)(5^{\beta+1}-1)\sigma(N)$$

$$+\frac{5}{3}(2^{\alpha+1}-1)(5^{\beta}-1)\sigma(N) - \frac{20}{3}(2^{\alpha-1}-1)(5^{\beta}-1)\sigma(N)$$

$$= 2(5^{\beta+1}-3)\sigma(N),$$

which is the first part of Theorem 7.1. Now suppose that $\alpha = 0$. Then

$$N(1, 1, 5, 5; n) = \frac{4}{3}\sigma(n) + \frac{20}{3}\sigma(n/5) + \frac{8}{3}c(n)$$

$$= \frac{1}{3}(5^{\beta+1}-1)\sigma(N) + \frac{5}{3}(5^{\beta}-1)\sigma(N) + \frac{8}{3}c(n)$$

$$= \frac{2}{3}(5^{\beta+1}-3)\sigma(N) + \frac{8}{3}c(n),$$

which is the second part of Theorem 7.1.

We conclude by giving a formula for $c(n)$ when $n \equiv 1 \pmod 2$.

Theorem 7.2. *Let* $n \in \mathbb{N}$ *be odd. Then*

$$c(n) = \sum_{\substack{(a,b,c,d)\in\mathbb{Z}^4 \\ 6n = a^2+9b^2+5c^2+45d^2 \\ a\equiv c\equiv 1 \pmod 3 \\ a+b\equiv c+d\equiv 1 \pmod 2}} (-1)^{b+d}.$$

Proof. It is known (see for example [18]) that

$$\prod_{n=1}^{\infty}(1-q^n)^2 = \sum_{\ell=0}^{\infty} p_2(\ell) q^{\ell},$$

where

$$p_2(\ell) = \sum_{\substack{(a,b)\in\mathbb{Z}^2 \\ 12\ell+1 = a^2+9b^2 \\ a\equiv 1 \pmod 3 \\ a+b\equiv 1 \pmod 2}} (-1)^{b}.$$

Thus

$$\sum_{n=0}^{\infty} c(2n+1)q^n = \sum_{\substack{n=1 \\ n \equiv 1 \pmod 2}}^{\infty} c(n)q^{(n-1)/2}$$

$$= \prod_{n=1}^{\infty} (1-q^n)^2(1-q^{5n})^2$$

$$= \left(\sum_{\ell=0}^{\infty} p_2(\ell)q^{\ell}\right)\left(\sum_{m=0}^{\infty} p_2(m)q^{5m}\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{\substack{\ell, m \geq 0 \\ \ell+5m=n}} p_2(l)p_2(m)\right)q^n$$

so that

$$c(2n+1) = \sum_{\substack{\ell, m \geq 0 \\ \ell+5m=n}} p_2(\ell)p_2(m)$$

$$= \sum_{\substack{\ell, m \geq 0 \\ \ell+5m=n}} \sum_{\substack{(a,b)\in\mathbb{Z}^2 \\ 12\ell+1=a^2+9b^2 \\ a\equiv 1 \pmod 3 \\ a+b\equiv 1 \pmod 2}} (-1)^b \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ 12m+1=c^2+9d^2 \\ c\equiv 1 \pmod 3 \\ c+d\equiv 1 \pmod 2}} (-1)^d$$

$$= \sum_{\substack{(a,b,c,d)\in\mathbb{Z}^4 \\ 12n+6=a^2+9b^2+5(c^2+9d^2) \\ a\equiv c\equiv 1 \pmod 3 \\ a+b\equiv c+d\equiv 1 \pmod 2}} (-1)^{b+d},$$

which is the asserted result.

It is known from the theory of modular forms that $c(n)$ is multiplicative, that

$$c(5^n) = (-1)^n, \quad n \in \mathbb{N} \cup \{0\},$$

and that for a prime $p \neq 2, 5$

$$c(p^{n+2}) = c(p)c(p^{n+1}) - pc(p^n), \quad n \in \mathbb{N} \cup \{0\},$$

see for example [15], [16] and [17].

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