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Jacobi's Identity and Representations of Integers by Certain Quaternary Quadratic Forms

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Abstract

Jacobi's identity

$$\prod_{n=1}^{\infty}(1-q^n)^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{n(n+1)}{2}} \quad (|q|<1)$$

is used to determine the number of representations of a positive integer by each of nineteen quaternary quadratic forms $x^2+by^2+cz^2+dt^2$ with $b,c,d \in \{1,2,4,8,16\}$ and $b \le c \le d$.

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1 Introduction

In a series of papers [7]- [25], Liouville gave without proof formulae for the number of representations of a positive integer n by each of the thirty-five quaternary quadratic forms $x^2+by^2+cz^2+dt^2$ with $b,c,d \in \{1,2,4,8,16\}$ and $b \le c \le d$. In [2] we determined the number of such representations for each of the ten forms $x^2+by^2+cz^2+dt^2$ with

$$\begin{aligned}(b,c,d) = \quad & (1,1,2),\ (1,1,8),\ (1,2,4),\ (1,4,8),\ (2,2,2),\\ & (2,2,8),\ (2,4,4),\ (2,8,8),\ (4,4,8),\ (8,8,8). \end{aligned} \tag{1.1}$$

In [3] we did the same for the six forms (among others) given by

$$(b,c,d) = \quad (1,1,1),\ (1,1,4),\ (1,2,2),\ (1,4,4),\ (2,2,4),\ (4,4,4). \tag{1.2}$$

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In this paper we use Jacobi's identity [6, p. 37]

$$\prod_{n=1}^{\infty}(1-q^n)^3=\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{n(n+1)}{2}},\ |q|<1, \tag{1.3}$$

to give the first proofs of Liouville's formulae for the number of representations of a positive integer n by the remaining nineteen quaternary quadratic forms $x^2+by^2+cz^2+dt^2$, that is, those given by

$$\begin{aligned}(b,c,d)= \quad & (1,1,16),\ (1,2,8),\ (1,2,16),\ (1,4,16),\ (1,8,8),\ (1,8,16),\\ & (1,16,16),\ (2,2,16),\ (2,4,8),\ (2,4,16),\ (2,8,16),\ (2,16,16),\\ & (4,4,16),\ (4,8,8),\ (4,8,16),\ (4,16,16),\ (8,8,16),\ (8,16,16),\\ & (16,16,16),\end{aligned} \tag{1.4}$$

see Theorems 4.1–4.19. The authors have treated representations of positive integers by other quaternary quadratic forms in [1] and [4].

2 Preliminary Results

We denote the sets of positive integers, nonnegative integers, and integers by $\mathbb{N},\ \mathbb{N}_0$, and $\mathbb{Z}$ respectively. We let $\mathbb{C}$ denote the field of complex numbers. Throughout this paper $q\in\mathbb{C}$ is such that $|q|<1$. We also set $\omega:=e^{2\pi i/8}$ so that $\omega=(1+i)/\sqrt{2},\ \omega^2=i,\ \omega^4=-1$ and $\omega^8=1$.

We define Ramanujan's theta functions φ and ψ by

$$\varphi(q):=\sum_{n=-\infty}^{\infty}q^{n^2} \tag{2.1}$$

and

$$\psi(q):=\sum_{n=0}^{\infty}q^{\frac{n(n+1)}{2}}, \tag{2.2}$$

see, e.g., [5, p. 6]. It is known that

$$\varphi(q)=\prod_{n=1}^{\infty}\frac{(1-q^{2n})^5}{(1-q^n)^2(1-q^{4n})^2}, \tag{2.3}$$

$$\varphi(-q)=\prod_{n=1}^{\infty}\frac{(1-q^n)^2}{(1-q^{2n})}, \tag{2.4}$$

and

$$\psi(q)=\prod_{n=1}^{\infty}\frac{(1-q^{2n})^2}{(1-q^n)}, \tag{2.5}$$

see, e.g., [5, p. 11]. We make use of the following properties of φ, all of which follow easily from (2.1). We remark that (2.6) is given in [5, p. 71] and (2.7) in [5, p. 72]. We have

$$\varphi(q)+\varphi(-q)=2\varphi(q^4), \tag{2.6}$$

$$\varphi^2(q)+\varphi^2(-q)=2\varphi^2(q^2), \tag{2.7}$$

$$\varphi(q)\varphi(-q)=\varphi^2(-q^2), \tag{2.8}$$

$$\sum_{n=-\infty}^{\infty} q^{(2n+1)^2}=\frac{1}{2}(\varphi(q)-\varphi(-q)), \tag{2.9}$$

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}=\varphi(-q), \tag{2.10}$$

$$\varphi(\pm iq)=\varphi(q^4)\pm\frac{i}{2}(\varphi(q)-\varphi(-q)), \tag{2.11}$$

and

$$\begin{cases} \varphi(\pm\omega q)=\varphi(-q^4)\pm\dfrac{\omega}{2}(\varphi(q)-\varphi(-q)), \\ \varphi(\pm\omega^3 q)=\varphi(-q^4)\pm\dfrac{\omega^3}{2}(\varphi(q)-\varphi(-q)). \end{cases} \tag{2.12}$$

We also need the following relationship between φ and ψ

$$\varphi(q)-\varphi(-q)=4q\psi(q^8), \tag{2.13}$$

which is a simple consequence of (2.2) and (2.9).

Theorem 2.1.

$$\sum_{\substack{m=-\infty \\ m\equiv 1 \,(\mathrm{mod}\ 2)}}^{\infty} (-1)^{\frac{m-1}{2}} m q^{m^2}=\frac{1}{2}(\varphi(q)-\varphi(-q))\varphi^2(-q^8).$$

Proof. By (2.5) and (2.13) we have

$$\frac{1}{2}(\varphi(q)-\varphi(-q))=2q\prod_{n=1}^{\infty}\frac{(1-q^{16n})^2}{(1-q^{8n})}. \tag{2.14}$$

Hence, appealing to (2.4), (2.14) and (1.3), we obtain

$$\tfrac{1}{2}(\varphi(q)-\varphi(-q))\varphi^2(-q^8) \quad = 2q\prod_{n=1}^{\infty}\frac{(1-q^{16n})^2}{(1-q^{8n})}\frac{(1-q^{8n})^4}{(1-q^{16n})^2}$$

$$
\begin{aligned}
&= 2q\prod_{n=1}^{\infty}\left(1-q^{8n}\right)^3 \\
&= 2q\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{4n(n+1)} \\
&= 2\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{(2n+1)^2} \\
&= \sum_{n=-\infty}^{\infty}(-1)^n(2n+1)q^{(2n+1)^2} \\
&= \sum_{\substack{m=-\infty \\ m\equiv 1\,(\mathrm{mod}\ 2)}}^{\infty}(-1)^{\frac{m-1}{2}}mq^{m^2},
\end{aligned}
$$

as asserted. □

For $m,n\in\mathbb{N}$ with $n\equiv 1$ (mod 2), we define

$$
K_m(n) := \sum_{\substack{(r,s)\in\mathbb{N}\times\mathbb{Z} \\ r\equiv 1\,(\mathrm{mod}\ 2) \\ n=r^2+2ms^2}}(-1)^{\frac{r-1}{2}}r = \frac{1}{2}\sum_{\substack{(r,s)\in\mathbb{Z}\times\mathbb{Z} \\ r\equiv 1\,(\mathrm{mod}\ 2) \\ n=r^2+2ms^2}}(-1)^{\frac{r-1}{2}}r. \tag{2.15}
$$

We make use of $K_1(n)$ and $K_2(n)$. The values of $K_1(n)$ and $K_2(n)$ $(n\in\mathbb{N},\ n \text{ odd},\ n\leq 49)$ are given in the table.

n	$K_1(n)$	$K_2(n)$	n	$K_1(n)$	$K_2(n)$	n	$K_1(n)$	$K_2(n)$
1	1	1	19	2	0	37	0	2
3	2	0	21	0	0	39	0	0
5	0	2	23	0	0	41	−6	10
7	0	0	25	5	−1	43	10	0
9	−1	−3	27	4	0	45	0	−6
11	−6	0	29	0	10	47	0	0
13	0	−6	31	0	0	49	−7	−7
15	0	0	33	12	0			
17	−6	2	35	0	0			

As $r^2+2s^2\equiv 0,1,2,3,4,6$ (mod 8) for $r,s\in\mathbb{Z}$, we see that if $n\in\mathbb{N}$ is such that $n\equiv 5,7$ (mod 8), then $n\neq r^2+2s^2$ for any $r,s\in\mathbb{Z}$. So

$$
K_1(n)=0,\ \text{if}\ n\equiv 5,7\ (\mathrm{mod}\ 8). \tag{2.16}
$$

Similarly, as $r^2+4s^2\equiv 0,1$ (mod 4) for $r,s\in\mathbb{Z}$, we see that if $n\in\mathbb{N}$ is such that $n\equiv 3$

(mod 4), then $n \neq r^2 + 4s^2$ for any $r, s \in \mathbb{Z}$. So

$$K_2(n) = 0, \text{ if } n \equiv 3 \pmod 4. \tag{2.17}$$

Theorem 2.2.

(i) $$\sum_{\substack{n=1 \\ n \equiv 1 \pmod 2}}^{\infty} K_1(n)q^n = \frac{1}{4}(\varphi(q) - \varphi(-q))\varphi(q^2)\varphi^2(-q^8).$$

(ii) $$\sum_{\substack{n=1 \\ n \equiv 1 \pmod 4}}^{\infty} K_1(n)q^n = \sum_{\substack{n=1 \\ n \equiv 1 \pmod 8}}^{\infty} K_1(n)q^n$$
$$= \frac{1}{4}(\varphi(q) - \varphi(-q))\varphi(q^8)\varphi^2(-q^8).$$

(iii) $$\sum_{\substack{n=1 \\ n \equiv 3 \pmod 4}}^{\infty} K_1(n)q^n = \sum_{\substack{n=1 \\ n \equiv 3 \pmod 8}}^{\infty} K_1(n)q^n$$
$$= \frac{1}{8}(\varphi(q) - \varphi(-q))(\varphi(q^2) - \varphi(-q^2))\varphi^2(-q^8).$$

Proof. Appealing to (2.16), (2.15), Theorem 2.1, and (2.1), we obtain

$$\sum_{\substack{n=1 \\ n \equiv 1 \pmod 2}}^{\infty} K_1(n)q^n = \frac{1}{2}\sum_{n=1}^{\infty} \sum_{\substack{(r,s) \in \mathbb{Z} \times \mathbb{Z} \\ r \equiv 1 \pmod 2 \\ n = r^2 + 2s^2}} (-1)^{\frac{r-1}{2}} r q^n$$
$$= \frac{1}{2}\Big(\sum_{r \in \mathbb{Z}} (-1)^{\frac{r-1}{2}} r q^{r^2}\Big)\Big(\sum_{s \in \mathbb{Z}} q^{2s^2}\Big)$$
$$= \frac{1}{4}(\varphi(q) - \varphi(-q))\varphi(q^2)\varphi^2(-q^8),$$

which is (i). Thus by (2.16) we have

$$\sum_{\substack{n=1 \\ n \equiv 1 \pmod 8}}^{\infty} K_1(n)q^n + \sum_{\substack{n=1 \\ n \equiv 3 \pmod 8}}^{\infty} K_1(n)q^n = \frac{1}{4}(\varphi(q) - \varphi(-q))\varphi(q^2)\varphi^2(-q^8).$$

Next we observe that if $n = r^2 + 2s^2$ with $r \equiv 1 \pmod 2$ then the Jacobi symbol

$$\left(\frac{2}{n}\right) = \left(\frac{2}{r^2 + 2s^2}\right) = \left(\frac{2}{1 + 2s^2}\right) = (-1)^s.$$

Hence, by (2.16), (2.15), Theorem 2.1, and (2.10), we deduce

$$\begin{aligned}
\sum_{\substack{n=1\\ n\equiv 1 \pmod 8}}^{\infty} K_1(n)q^n - \sum_{\substack{n=1\\ n\equiv 3 \pmod 8}}^{\infty} K_1(n)q^n
&= \sum_{\substack{n=1\\ n\equiv 1 \pmod 2}}^{\infty} \left(\frac{2}{n}\right) K_1(n)q^n \\
&= \sum_{n=1}^{\infty} \left(\frac{2}{n}\right) \Big(\frac{1}{2} \sum_{\substack{(r,s)\in\mathbb{Z}\times\mathbb{Z}\\ r\equiv 1 \pmod 2\\ n=r^2+2s^2}} (-1)^{\frac{r-1}{2}} r\Big) q^n \\
&= \frac{1}{2} \sum_{n=1}^{\infty} \sum_{\substack{(r,s)\in\mathbb{Z}\times\mathbb{Z}\\ r\equiv 1 \pmod 2\\ n=r^2+2s^2}} (-1)^{\frac{r-1}{2}} r (-1)^s q^{r^2+2s^2} \\
&= \frac{1}{2} \Big(\sum_{\substack{r\in\mathbb{Z}\\ r\equiv 1 \pmod 2}} (-1)^{\frac{r-1}{2}} r q^{r^2} \Big) \Big(\sum_{s\in\mathbb{Z}} (-1)^s q^{2s^2} \Big) \\
&= \frac{1}{4} (\varphi(q) - \varphi(-q)) \varphi(-q^2) \varphi^2(-q^8).
\end{aligned}$$

Adding and subtracting these two results, and appealing to (2.6) and (2.16), we obtain (ii) and (iii). □

Theorem 2.3.

$$\text{(i)} \quad \sum_{\substack{n=1\\ n\equiv 1 \pmod 2}}^{\infty} K_2(n)q^n = \sum_{\substack{n=1\\ n\equiv 1 \pmod 4}}^{\infty} K_2(n)q^n = \frac{1}{4}(\varphi(q) - \varphi(-q))\varphi(q^4)\varphi^2(-q^8).$$

$$\text{(ii)} \quad \sum_{\substack{n=1\\ n\equiv 1 \pmod 8}}^{\infty} K_2(n)q^n = \frac{1}{4}(\varphi(q) - \varphi(-q))\varphi^2(-q^8)\varphi(q^{16}).$$

$$\text{(iii)} \quad \sum_{\substack{n=1\\ n\equiv 5 \pmod 8}}^{\infty} K_2(n)q^n = \frac{1}{8}(\varphi(q) - \varphi(-q))(\varphi(q^4) - \varphi(-q^4))\varphi^2(-q^8).$$

$$\text{(iv)} \qquad \sum_{n=1}^{\infty} \left(\frac{2}{n}\right) K_2(n) q^n = \frac{1}{4}(\varphi(q) - \varphi(-q))\varphi(-q^4)\varphi^2(-q^8).$$

Proof. The proof is similar to that of Theorem 2.2. □

For $n \in \mathbb{N}$ we define

$$\sigma(n) := \sum_{d|n} d,$$

where d runs through the positive integers dividing n. If $n \notin \mathbb{N}$ we set $\sigma(n) = 0$.

The Eisenstein series E_2 is defined by

$$E_2(q) := 1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n, \tag{2.18}$$

see for example [5, p. 87].

Theorem 2.4.

$$\text{(i)} \qquad \sum_{\substack{n=1 \\ n \equiv 1 \pmod 2}}^{\infty} \sigma(n) q^n = \frac{1}{4}(\varphi(q) - \varphi(-q))\varphi^2(q^2)\varphi(q^4).$$

$$\text{(ii)} \qquad \sum_{\substack{n=1 \\ n \equiv 1 \pmod 4}}^{\infty} \sigma(n) q^n = \frac{1}{4}(\varphi(q) - \varphi(-q))\varphi^3(q^4).$$

$$\text{(iii)} \qquad \sum_{\substack{n=1 \\ n \equiv 3 \pmod 4}}^{\infty} \sigma(n) q^n = \frac{1}{4}(\varphi(q) - \varphi(-q))(\varphi(q^2) - \varphi(-q^2))\varphi(q^4)\varphi(q^8).$$

$$\text{(iv)} \qquad \sum_{\substack{n=1 \\ n \equiv 1 \pmod 8}}^{\infty} \sigma(n) q^n = \frac{1}{8}(\varphi(q) - \varphi(-q))(\varphi^3(q^4) + \varphi^3(-q^4)).$$

$$\text{(v)} \qquad \sum_{\substack{n=1 \\ n \equiv 3 \pmod 8}}^{\infty} \sigma(n) q^n = \frac{1}{4}(\varphi(q) - \varphi(-q))(\varphi(q^2) - \varphi(-q^2))\varphi(q^8)\varphi(q^{16}).$$

$$\text{(vi)} \qquad \sum_{\substack{n=1 \\ n \equiv 5 \pmod 8}}^{\infty} \sigma(n) q^n = \frac{1}{8}(\varphi(q) - \varphi(-q))(\varphi^3(q^4) - \varphi^3(-q^4)).$$

$$\text{(vii)} \qquad \sum_{\substack{n=1 \\ n \equiv 7 \pmod 8}}^{\infty} \sigma(n) q^n = \frac{1}{8}(\varphi(q) - \varphi(-q))(\varphi(q^2) - \varphi(-q^2))(\varphi(q^4) - \varphi(-q^4))\varphi(q^8).$$

Proof. It is known that

$$E_2(q) = (1-5x)z^2 + 12x(1-x)z\frac{dz}{dx} \tag{2.19}$$

and

$$E_2(-q) = (1-2x)z^2 + 12x(1-x)z\frac{dz}{dx}, \tag{2.20}$$

where

$$x = 1 - \frac{\varphi^4(-q)}{\varphi^4(q)}, \;\; z = \varphi^2(q), \tag{2.21}$$

see for example [5, Section 5.4]. Hence, by (2.18)–(2.21), we obtain

$$\begin{aligned}
\sum_{\substack{n=1 \\ n \equiv 1 \pmod 2}}^{\infty} \sigma(n)q^n &= \frac{1}{2}\sum_{n=1}^{\infty}\sigma(n)q^n - \frac{1}{2}\sum_{n=1}^{\infty}\sigma(n)(-q)^n \\
&= \frac{1}{48}(E_2(-q) - E_2(q)) \\
&= \frac{1}{16}xz^2 \\
&= \frac{1}{16}(\varphi^4(q) - \varphi^4(-q)).
\end{aligned}$$

Now, by (2.6) and (2.7), we obtain

$$\begin{aligned}
\varphi^4(q) - \varphi^4(-q) &= (\varphi(q)+\varphi(-q))(\varphi(q)-\varphi(-q))(\varphi^2(q)+\varphi^2(-q)) \\
&= 2\varphi(q^4)(\varphi(q)-\varphi(-q))2\varphi^2(q^2) \\
&= 4(\varphi(q)-\varphi(-q))\varphi^2(q^2)\varphi(q^4).
\end{aligned}$$

This completes the proof of part (i).

Next, by part (i), we have

$$\begin{aligned}
&\sum_{\substack{n=1 \\ n \equiv 1 \pmod 4}}^{\infty} \sigma(n)q^n - \sum_{\substack{n=1 \\ n \equiv 3 \pmod 4}}^{\infty} \sigma(n)q^n \\
&= \frac{1}{i}\sum_{\substack{n=1 \\ n \equiv 1 \pmod 2}}^{\infty} \sigma(n)(iq)^n \\
&= \frac{1}{4i}(\varphi(iq) - \varphi(-iq))\varphi^2(-q^2)\varphi(q^4),
\end{aligned}$$

that is by (2.11)

$$\sum_{\substack{n=1 \\ n\equiv 1\,(\mathrm{mod}\ 4)}}^{\infty} \sigma(n)q^n - \sum_{\substack{n=1 \\ n\equiv 3\,(\mathrm{mod}\ 4)}}^{\infty} \sigma(n)q^n = \frac{1}{4}(\varphi(q)-\varphi(-q))\varphi^2(-q^2)\varphi(q^4). \quad (2.22)$$

Also, by part (i), we have

$$\sum_{\substack{n=1 \\ n\equiv 1\,(\mathrm{mod}\ 4)}}^{\infty} \sigma(n)q^n + \sum_{\substack{n=1 \\ n\equiv 3\,(\mathrm{mod}\ 4)}}^{\infty} \sigma(n)q^n = \frac{1}{4}(\varphi(q)-\varphi(-q))\varphi^2(q^2)\varphi(q^4). \quad (2.23)$$

Adding and subtracting (2.22) and (2.23), and using (2.6), we obtain parts (ii) and (iii).

Appealing to part (ii) and (2.12), we obtain

$$\begin{aligned}
\omega\Big(\sum_{\substack{n=1 \\ n\equiv 1\,(\mathrm{mod}\ 8)}}^{\infty} \sigma(n)q^n - \sum_{\substack{n=1 \\ n\equiv 5\,(\mathrm{mod}\ 8)}}^{\infty} \sigma(n)q^n\Big)
&= \sum_{\substack{n=1 \\ n\equiv 1\,(\mathrm{mod}\ 4)}}^{\infty} \sigma(n)(\omega q)^n \\
&= \frac{1}{4}(\varphi(\omega q)-\varphi(-\omega q))\varphi^3(-q^4) \\
&= \frac{1}{4}\omega(\varphi(q)-\varphi(-q))\varphi^3(-q^4)
\end{aligned}$$

so that

$$\sum_{\substack{n=1 \\ n\equiv 1\,(\mathrm{mod}\ 8)}}^{\infty} \sigma(n)q^n - \sum_{\substack{n=1 \\ n\equiv 5\,(\mathrm{mod}\ 8)}}^{\infty} \sigma(n)q^n = \frac{1}{4}(\varphi(q)-\varphi(-q))\varphi^3(-q^4). \quad (2.24)$$

Again by part (ii) we have

$$\sum_{\substack{n=1 \\ n\equiv 1\,(\mathrm{mod}\ 8)}}^{\infty} \sigma(n)q^n + \sum_{\substack{n=1 \\ n\equiv 5\,(\mathrm{mod}\ 8)}}^{\infty} \sigma(n)q^n = \frac{1}{4}(\varphi(q)-\varphi(-q))\varphi^3(q^4). \quad (2.25)$$

Adding and subtracting (2.24) and (2.25), we obtain parts (iv) and (vi) of the theorem.

Parts (v) and (vii) follow in a similar fashion from part (iii). □

Finally in this section we define

$$S(n) := \sum_{d|n} \frac{n}{d}\left(\frac{2}{d}\right),\ n\in\mathbb{N}, \quad (2.26)$$

where $\left(\frac{2}{d}\right)$ is the Kronecker symbol. It was shown in [2, Theorem 4.2(b)] that

$$\sum_{n=1}^{\infty} S(n)q^n = \frac{1}{2}\varphi^2(q)\varphi(q^2)\varphi(q^4) - \frac{1}{2}\varphi(q)\varphi(q^2)\varphi^2(q^4)$$

so by (2.6) we have

$$\sum_{n=1}^{\infty} S(n)q^n = \frac{1}{4}(\varphi(q) - \varphi(-q))\varphi(q)\varphi(q^2)\varphi(q^4). \tag{2.27}$$

It was also noted in [2, Theorem 4.1(b)] that if $n = 2^\alpha N,\ n \in \mathbb{N},\ \alpha \in \mathbb{N}_0,\ N \in \mathbb{N}$ and $N \equiv 1 \pmod 2$ then

$$S(n) = 2^\alpha S(N). \tag{2.28}$$

Theorem 2.5.

(i) $$\sum_{\substack{n=1 \\ n \equiv 1 \pmod 2}}^{\infty} S(n)q^n = \frac{1}{4}(\varphi(q) - \varphi(-q))\varphi(q^2)\varphi^2(q^4).$$

(ii) $$\sum_{\substack{n=1 \\ n \equiv 2 \pmod 4}}^{\infty} S(n)q^n = \frac{1}{2}(\varphi(q^2) - \varphi(-q^2))\varphi(q^4)\varphi^2(q^8).$$

(iii) $$\sum_{\substack{n=1 \\ n \equiv 1 \pmod 4}}^{\infty} S(n)q^n = \frac{1}{4}(\varphi(q) - \varphi(-q))\varphi^2(q^4)\varphi(q^8).$$

(iv) $$\sum_{\substack{n=1 \\ n \equiv 3 \pmod 4}}^{\infty} S(n)q^n = \frac{1}{8}(\varphi(q) - \varphi(-q))(\varphi(q^2) - \varphi(-q^2))\varphi^2(q^4).$$

(v) $$\sum_{\substack{n=1 \\ n \equiv 1 \pmod 8}}^{\infty} S(n)q^n = \frac{1}{4}(\varphi(q) - \varphi(-q))\varphi^3(q^8).$$

(vi) $$\sum_{\substack{n=1 \\ n \equiv 3 \pmod 8}}^{\infty} S(n)q^n = \frac{1}{8}(\varphi(q) - \varphi(-q))(\varphi(q^2) - \varphi(-q^2))\varphi^2(q^8).$$

(vii) $$\sum_{\substack{n=1 \\ n \equiv 5 \pmod 8}}^{\infty} S(n)q^n = \frac{1}{4}(\varphi(q) - \varphi(-q))(\varphi(q^4) - \varphi(-q^4))\varphi(q^8)\varphi(q^{16}).$$

(viii) $$\sum_{\substack{n=1\\ n\equiv 7\,(\mathrm{mod}\ 8)}}^{\infty} S(n)q^n = \frac{1}{8}(\varphi(q)-\varphi(-q))(\varphi(q^2)-\varphi(-q^2)) \times(\varphi(q^4)-\varphi(-q^4))\varphi(q^{16}).$$

Proof. Appealing to (2.27) and (2.6), we obtain

$$\begin{aligned}\sum_{\substack{n=1\\ n\equiv 1\,(\mathrm{mod}\ 2)}}^{\infty} S(n)q^n &= \frac{1}{2}\sum_{n=1}^{\infty} S(n)(1-(-1)^n)q^n \\
&= \frac{1}{2}\sum_{n=1}^{\infty} S(n)q^n - \frac{1}{2}\sum_{n=1}^{\infty} S(n)(-q)^n \\
&= \frac{1}{8}(\varphi(q)-\varphi(-q))\varphi(q)\varphi(q^2)\varphi(q^4) \\
&\quad -\frac{1}{8}(\varphi(-q)-\varphi(q))\varphi(-q)\varphi(q^2)\varphi(q^4) \\
&= \frac{1}{8}(\varphi(q)-\varphi(-q))\varphi(q)\varphi(q^2)\varphi(q^4) \\
&\quad +\frac{1}{8}(\varphi(q)-\varphi(-q))\varphi(-q)\varphi(q^2)\varphi(q^4) \\
&= \frac{1}{8}(\varphi(q)-\varphi(-q))(\varphi(q)+\varphi(-q))\varphi(q^2)\varphi(q^4) \\
&= \frac{1}{4}(\varphi(q)-\varphi(-q))\varphi(q^2)\varphi^2(q^4),\end{aligned}$$

which is part (i).

For $n \equiv 2$ (mod 4) we can write $n = 2N$, where $N \equiv 1$ (mod 2). Then, by (2.28), we have $S(n) = 2S(N)$. Hence, by part (i), we obtain

$$\begin{aligned}\sum_{\substack{n=1\\ n\equiv 2\,(\mathrm{mod}\ 4)}}^{\infty} S(n)q^n &= 2\sum_{\substack{N=1\\ N\equiv 1\,(\mathrm{mod}\ 2)}}^{\infty} S(N)q^{2N} \\
&= \frac{1}{2}(\varphi(q^2)-\varphi(-q^2))\varphi(q^4)\varphi^2(q^8),\end{aligned}$$

which is part (ii).

Appealing to Theorem 2.7, we obtain

$$\sum_{\substack{n=1\\ n\equiv 1\,(\mathrm{mod}\ 4)}}^{\infty} S(n)q^n - \sum_{\substack{n=1\\ n\equiv 3\,(\mathrm{mod}\ 4)}}^{\infty} S(n)q^n$$

$$= \frac{1}{i} \sum_{\substack{n=1 \\ n \equiv 1 \pmod 2}}^{\infty} S(n)(iq)^n$$

$$= \frac{1}{4i}(\varphi(iq) - \varphi(-iq))\varphi(-q^2)\varphi^2(q^4),$$

that is by (2.11)

$$\sum_{\substack{n=1 \\ n \equiv 1 \pmod 4}}^{\infty} S(n)q^n - \sum_{\substack{n=1 \\ n \equiv 3 \pmod 4}}^{\infty} S(n)q^n = \frac{1}{4}(\varphi(q) - \varphi(-q))\varphi(-q^2)\varphi^2(q^4). \tag{2.29}$$

Also, by part (i), we have

$$\sum_{\substack{n=1 \\ n \equiv 1 \pmod 4}}^{\infty} S(n)q^n + \sum_{\substack{n=1 \\ n \equiv 3 \pmod 4}}^{\infty} S(n)q^n = \frac{1}{4}(\varphi(q) - \varphi(-q))\varphi(q^2)\varphi^2(q^4). \tag{2.30}$$

Adding (2.29) and (2.30), we obtain by (2.6)

$$2 \sum_{\substack{n=1 \\ n \equiv 1 \pmod 4}}^{\infty} S(n)q^n = \frac{1}{4}(\varphi(q) - \varphi(-q))(\varphi(q^2) + \varphi(-q^2))\varphi^2(q^4)$$

$$= \frac{1}{2}(\varphi(q) - \varphi(-q))\varphi(q^8)\varphi^2(q^4),$$

which gives part (iii). Subtracting (2.29) from (2.30), we deduce

$$2 \sum_{\substack{n=1 \\ n \equiv 3 \pmod 4}}^{\infty} S(n)q^n = \frac{1}{4}(\varphi(q) - \varphi(-q))(\varphi(q^2) - \varphi(-q^2))\varphi^2(q^4),$$

which gives part (iv).

By part (iii) we obtain

$$\sum_{\substack{n=1 \\ n \equiv 1 \pmod 8}}^{\infty} S(n)q^n - \sum_{\substack{n=1 \\ n \equiv 5 \pmod 8}}^{\infty} S(n)q^n$$

$$= \frac{1}{\omega} \sum_{\substack{n=1 \\ n \equiv 1 \,(\mathrm{mod}\ 4)}}^{\infty} S(n)(\omega q)^n$$

$$= \frac{1}{4\omega}(\varphi(\omega q) - \varphi(-\omega q))\varphi^2(-q^4)\varphi(q^8),$$

that is

$$\sum_{\substack{n=1 \\ n \equiv 1 \,(\mathrm{mod}\ 8)}}^{\infty} S(n)q^n - \sum_{\substack{n=1 \\ n \equiv 5 \,(\mathrm{mod}\ 8)}}^{\infty} S(n)q^n$$

$$= \frac{1}{4}(\varphi(q) - \varphi(-q))\varphi^2(-q^4)\varphi(q^8). \tag{2.31}$$

Also, by part (iii), we have

$$\sum_{\substack{n=1 \\ n \equiv 1 \,(\mathrm{mod}\ 8)}}^{\infty} S(n)q^n + \sum_{\substack{n=1 \\ n \equiv 5 \,(\mathrm{mod}\ 8)}}^{\infty} S(n)q^n = \frac{1}{4}(\varphi(q) - \varphi(-q))\varphi^2(q^4)\varphi(q^8). \tag{2.32}$$

Adding (2.31) and (2.32), we obtain by (2.7)

$$2 \sum_{\substack{n=1 \\ n \equiv 1 \,(\mathrm{mod}\ 8)}}^{\infty} S(n)q^n = \frac{1}{4}(\varphi(q) - \varphi(-q))(\varphi^2(q^4) + \varphi^2(-q^4))\varphi(q^8)$$

$$= \frac{1}{2}(\varphi(q) - \varphi(-q))\varphi^3(q^8),$$

from which part (v) follows. Subtracting (2.31) from (2.32), we deduce

$$2 \sum_{\substack{n=1 \\ n \equiv 5 \,(\mathrm{mod}\ 8)}}^{\infty} S(n)q^n = \frac{1}{4}(\varphi(q) - \varphi(-q))(\varphi^2(q^4) - \varphi^2(-q^4))\varphi(q^8)$$

$$= \frac{1}{4}(\varphi(q) - \varphi(-q))(\varphi(q^4) - \varphi(-q^4))(\varphi(q^4) + \varphi(-q^4))\varphi(q^8)$$
$$= \frac{1}{2}(\varphi(q) - \varphi(-q))(\varphi(q^4) - \varphi(-q^4))\varphi(q^8)\varphi(q^{16}),$$

which gives part (vii).

Parts (vi) and (viii) can be proved similarly from part (iv). □

3 Theta Function Identities

In this section we prove a number of identities involving φ. These identities will be used in the proofs of Theorems 4.1-4.19.

Theorem 3.1.

(i) $\varphi(q)\varphi(q^2)\varphi(q^4) = \frac{1}{2}\varphi^2(q)\varphi(q^2) + \frac{1}{2}\varphi(-q^2)\varphi^2(-q^4)$.

(ii) $\varphi(q)\varphi^2(q^2) = \frac{1}{2}\varphi^3(q) + \frac{1}{2}\varphi(-q)\varphi^2(-q^2)$.

(iii) $\varphi(q)\varphi^2(q^4) = \frac{1}{4}\varphi^3(q) + \frac{1}{2}\varphi(q)\varphi^2(-q^2) + \frac{1}{4}\varphi(-q)\varphi^2(-q^2)$.

(iv) $\varphi(q)\varphi(q^4) = \frac{1}{2}\varphi^2(q) + \frac{1}{2}\varphi^2(-q^2)$.

(v) $\varphi^2(q)\varphi(q^8) = \frac{1}{2}\varphi^3(q^2) + \frac{1}{2}\varphi(q^2)\varphi^2(-q^4) + (\varphi(q) - \varphi(-q))\varphi(q^4)\varphi(q^8)$.

(vi) $$(\varphi^2(q) + \varphi(q)\varphi(-q) + \varphi^2(-q))\varphi(q^{16}) \\ = \frac{3}{2}\varphi^3(q^4) + (\varphi(q^2) - \varphi(-q^2))\varphi(q^8)\varphi(q^{16}) + \frac{3}{2}\varphi(q^4)\varphi^2(-q^8).$$

(vii) $\varphi^2(q^2)\varphi(q^4) = \frac{1}{4}\varphi^3(q) + \frac{1}{4}\varphi^3(-q) + \frac{1}{2}\varphi^2(-q^2)\varphi(q^4)$.

(viii) $\varphi^3(q^4) = \frac{1}{8}(\varphi^3(q) + \varphi^3(-q)) + \frac{3}{4}\varphi^2(-q^2)\varphi(q^4)$.

(ix) $$\varphi(q)\varphi(q^4)\varphi(q^8) = \frac{1}{4}(\varphi^3(q^2) + \varphi^3(-q^2)) \\ + \frac{1}{2}\varphi^2(-q^4)\varphi(q^8) + \frac{1}{2}(\varphi(q) - \varphi(-q))\varphi(q^4)\varphi(q^8).$$

(x) $$\varphi(q)\varphi^2(q^8) = \frac{1}{2}\varphi^3(q^4) + \frac{1}{2}\varphi(q^4)\varphi^2(-q^4) \\ + \frac{1}{4}(\varphi(q) - \varphi(-q))\varphi^2(q^4) + \frac{1}{4}(\varphi(q) - \varphi(-q))\varphi^2(-q^4).$$

Proof. We just prove part (vi) as the rest can be proved similarly. For brevity we set

$$a = \varphi(q),\ b = \varphi(-q),\ c = \varphi(q^2),\ d = \varphi(-q^2),\ e = \varphi(q^4), \\ f = \varphi(-q^4),\ g = \varphi(q^8),\ h = \varphi(-q^8),\ j = \varphi(q^{16}).$$

From (2.6)-(2.8), we obtain

$$a + b = 2e,\ c + d = 2g,\ e + f = 2j, \\ a^2 + b^2 = 2c^2,\ c^2 + d^2 = 2e^2,\ e^2 + f^2 = 2g^2, \\ ab = d^2,\ cd = f^2,\ ef = h^2.$$

We have

$$\frac{3}{2}\varphi^3(q^4) + (\varphi(q^2) - \varphi(-q^2))\varphi(q^8)\varphi(q^{16}) + \frac{3}{2}\varphi(q^4)\varphi^2(-q^8)$$

$$
\begin{aligned}
&= \frac{3}{2}e^3 + (c-d)gj + \frac{3}{2}eh^2 \\
&= \frac{3}{2}e^3 + (c-d)gj + \frac{3}{2}e^2 f \\
&= \frac{3}{2}e^2(e+f) + (c-d)gj \\
&= j(3e^2 + (c-d)g) \\
&= j(3e^2 + \frac{1}{2}(c^2-d^2)) \\
&= j(\frac{3}{4}(a+b)^2 + \frac{1}{4}(a^2+b^2) - \frac{1}{2}ab) \\
&= (a^2+ab+b^2)j \\
&= (\varphi^2(q) + \varphi(q)\varphi(-q) + \varphi^2(-q))\varphi(q^{16}).
\end{aligned}
$$

□

4 Representations by Quaternary Quadratic Forms

For $n \in \mathbb{N}_0$ and $A, B, C, D \in \mathbb{N}$ we set

$$N(A,B,C,D;n) = \text{card}\{(x,y,z,t) \in \mathbb{Z}^4 \mid n = Ax^2 + By^2 + Cz^2 + Dt^2\}. \quad (4.1)$$

Clearly

$$N(A,B,C,D;0) = 1. \quad (4.2)$$

Also if A_1, B_1, C_1, D_1 is any permutation of A, B, C, D, we have

$$N(A_1,B_1,C_1,D_1;n) = N(A,B,C,D;n),\ n \in \mathbb{N}_0. \quad (4.3)$$

Further

$$\sum_{n=0}^{\infty} N(A,B,C,D;n)q^n = \varphi(q^A)\varphi(q^B)\varphi(q^C)\varphi(q^D). \quad (4.4)$$

Moreover, if $B \equiv C \equiv D \equiv 0 \pmod 2$ then any $(x,y,z,t) \in \mathbb{Z}^4$ satisfying $n = x^2 + By^2 + Cz^2 + Dt^2$ with $n \equiv 1 \pmod 2$ has $x \equiv 1 \pmod 2$ and so by (2.9) we have

$$\sum_{\substack{n=0 \\ n \equiv 1 \,(\text{mod } 2)}}^{\infty} N(1,B,C,D;n)q^n = \frac{1}{2}(\varphi(q) - \varphi(-q))\varphi(q^B)\varphi(q^C)\varphi(q^D), \quad \text{if } B \equiv C \equiv D \equiv 0 \,(\text{mod } 2). \quad (4.5)$$

Similarly

$$\sum_{\substack{n=0 \\ n \equiv 1 \,(\text{mod } 4)}}^{\infty} N(1,B,C,D;n)q^n = \frac{1}{2}(\varphi(q) - \varphi(-q))\varphi(q^B)\varphi(q^C)\varphi(q^D), \quad \text{if } B \equiv C \equiv D \equiv 0 \,(\text{mod } 4). \quad (4.6)$$

and

$$\sum_{\substack{n=0 \\ n\equiv 1 \,(\mathrm{mod}\ 8)}}^{\infty} N(1,B,C,D;n)q^n = \frac{1}{2}(\varphi(q)-\varphi(-q))\varphi(q^B)\varphi(q^C)\varphi(q^D), \quad \text{if } B \equiv C \equiv D \equiv 0 \,(\mathrm{mod}\ 8). \tag{4.7}$$

Next we give some elementary properties of $N(1,b,c,d;n)$ $(n \in \mathbb{N},\ b,c,d \in \{1,2,4,8,16\})$.

Proposition 4.1. *Let* $b,c,d \in \{1,2,4,8,16\}$. *Let* $n \in \mathbb{N}$.

(i) *If* $d \equiv 0$ (mod 4) *then*

$$N(1,1,1,d;n) = N(1,1,1,d/4;n/4),\ \textit{if}\ n \equiv 0 \,(\mathrm{mod}\ 4).$$

(ii) *If* $d \equiv 0$ (mod 4) *then*

$$N(1,1,1,d;n) = 3N(1,1,2,d/2;n/2),\ \textit{if}\ n \equiv 2 \,(\mathrm{mod}\ 4).$$

(iii) *If* $c \equiv d \equiv 0$ (mod 2) *then*

$$N(1,1,c,d;n) = N(1,1,c/2,d/2;n/2),\ \textit{if}\ n \equiv 0 \,(\mathrm{mod}\ 2).$$

(iv) *If* $c \equiv d \equiv 0$ (mod 2) *then*

$$N(1,1,c,d;n) = 2N(1,4,c,d;n),\ \textit{if}\ n \equiv 1 \,(\mathrm{mod}\ 2).$$

(v) *If* $c \equiv d \equiv 0$ (mod 4) *then*

$$N(1,1,c,d;n) = 0,\ \textit{if}\ n \equiv 3 \,(\mathrm{mod}\ 4).$$

(vi) *If* $c \equiv d \equiv 0$ (mod 8) *then*

$$N(1,1,c,d;n) = 0,\ \textit{if}\ n \equiv 3,6,7 \,(\mathrm{mod}\ 8).$$

(vii) *If* $c \equiv d \equiv 0$ (mod 2) *then*

$$N(1,2,c,d;n) = N(1,2,c/2,d/2;n/2),\ \textit{if}\ n \equiv 0 \,(\mathrm{mod}\ 2).$$

(viii) *If* $c \equiv d \equiv 0$ (mod 8) *then*

$$N(1,2,c,d;n) = 0,\ \textit{if}\ n \equiv 5,7 \,(\mathrm{mod}\ 8).$$

(ix) *If* $b \equiv c \equiv d \equiv 0$ (mod 2) *then*

$$N(1,b,c,d;n) = N(2,b/2,c/2,d/2;n/2),\ \textit{if}\ n \equiv 0 \,(\mathrm{mod}\ 2).$$

(x) *If* $b \equiv c \equiv d \equiv 0 \pmod 4$ *then*

$$N(1,b,c,d;n) = N(1,b/4,c/4,d/4;n/4), \text{ if } n \equiv 0 \pmod 4.$$

(xi) *If* $b \equiv c \equiv d \equiv 0 \pmod 4$ *then*

$$N(1,b,c,d;n) = 0, \text{ if } n \equiv 2,3 \pmod 4.$$

(xii) *If* $b \equiv c \equiv d \equiv 0 \pmod 8$ *then*

$$N(1,b,c,d;n) = 0, \text{ if } n \equiv 2,3,5,6,7 \pmod 8.$$

Proof. We just give the proof of part (i) as the remaining parts can be proved in a similar (or easier) manner.

Let $n \equiv d \equiv 0 \pmod 4$. Set

$$S = \{(x,y,z,t) \in \mathbb{Z}^4 \mid n = x^2 + y^2 + z^2 + dt^2\}$$

and

$$T = \{(x,y,z,t) \in \mathbb{Z}^4 \mid \frac{n}{4} = x^2 + y^2 + z^2 + \frac{d}{4}t^2\}.$$

Define $\lambda : T \to S$ by

$$\lambda((x,y,z,t)) = (2x,2y,2z,t).$$

Clearly λ is injective. Suppose $(x,y,z,t) \in S$. Then $n = x^2 + y^2 + z^2 + dt^2$. As $n \equiv d \equiv 0 \pmod 4$ we have $x^2 + y^2 + z^2 \equiv 0 \pmod 4$ so $x \equiv y \equiv z \equiv 0 \pmod 2$. Then $(x/2, y/2, z/2, t) \in T$ and $\lambda((x/2, y/2, z/2, t)) = (x,y,z,t)$. Hence λ is surjective. Thus λ is a bijection and

$$N(1,1,1,d;n) = \text{card}\, S = \text{card}\, \lambda S = \text{card}\, T = N(1,1,1,d/4;n/4).$$

□

The results of the next proposition were proved in [3].

Proposition 4.2. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha N$, *where* $\alpha \in \mathbb{N}_0,\ N \in \mathbb{N}$ *and* $N \equiv 1 \pmod 2$. *Then*

$$\text{(i)}\quad N(1,1,1,4;n) = \begin{cases} 6\sigma(N), & \text{if } \alpha = 0,\ N \equiv 1 \pmod 4, \\ 2\sigma(N), & \text{if } \alpha = 0,\ N \equiv 3 \pmod 4, \\ 12\sigma(N), & \text{if } \alpha = 1, \\ 8\sigma(N), & \text{if } \alpha = 2, \\ 24\sigma(N), & \text{if } \alpha \geq 3, \end{cases}$$

$$\text{(ii)}\quad N(1,1,2,2;n)=\begin{cases} 4\sigma(N), & \textit{if } \alpha=0,\\ 8\sigma(N), & \textit{if } \alpha=1,\\ 24\sigma(N), & \textit{if } \alpha\geq 2,\end{cases}$$

$$\text{(iii)}\quad N(1,1,4,4;n)=\begin{cases} 4\sigma(N), & \textit{if } \alpha=0,\ N\equiv 1 \,(\mathrm{mod}\ 4),\\ 0, & \textit{if } \alpha=0,\ N\equiv 3 \,(\mathrm{mod}\ 4),\\ 4\sigma(N), & \textit{if } \alpha=1,\\ 8\sigma(N), & \textit{if } \alpha=2,\\ 24\sigma(N), & \textit{if } \alpha\geq 3,\end{cases}$$

$$\text{(iv)}\quad N(1,2,2,4;n)=\begin{cases} 2\sigma(N), & \textit{if } \alpha=0,\\ 4\sigma(N), & \textit{if } \alpha=1,\\ 8\sigma(N), & \textit{if } \alpha=2,\\ 24\sigma(N), & \textit{if } \alpha\geq 3,\end{cases}$$

$$\text{(v)}\quad N(1,4,4,4;n)=\begin{cases} 2\sigma(N), & \textit{if } \alpha=0,\ N\equiv 1 \,(\mathrm{mod}\ 4),\\ 0, & \textit{if } \alpha=0,\ N\equiv 3 \,(\mathrm{mod}\ 4),\\ 0, & \textit{if } \alpha=1,\\ 8\sigma(N), & \textit{if } \alpha=2,\\ 24\sigma(N), & \textit{if } \alpha\geq 3.\end{cases}$$

The results of the final proposition of this section were proved in [2].

Proposition 4.3. *Let* $n\in\mathbb{N}$. *Set* $n=2^{\alpha}N$, *where* $\alpha\in\mathbb{N}_0,\ N\in\mathbb{N}$ *and* $N\equiv 1$ (mod 2). *Then*

$$\text{(i)}\quad N(1,1,1,8;n)=\begin{cases} 6S(N), & \textit{if } \alpha=0,\ N\equiv 1 \,(\mathrm{mod}\ 4),\\ 4S(N), & \textit{if } \alpha=0,\ N\equiv 3 \,(\mathrm{mod}\ 8),\\ 0, & \textit{if } \alpha=0,\ N\equiv 7 \,(\mathrm{mod}\ 8),\\ 12S(N), & \textit{if } \alpha=1,\\ \left(2^{\alpha+1}-2\left(\dfrac{2}{N}\right)\right)S(N), & \textit{if } \alpha\geq 2.\end{cases}$$

$$\text{(ii)}\quad N(1,1,2,4;n)=\begin{cases} 4S(N), & \textit{if } \alpha=0,\\ \left(2^{\alpha+2}-2\left(\dfrac{2}{N}\right)\right)S(N), & \textit{if } \alpha\geq 1.\end{cases}$$

$$\text{(iii)} \quad N(1,1,4,8;n) = \begin{cases} 4S(N), & \textit{if } \alpha = 0,\ N \equiv 1 \pmod 4, \\ 0, & \textit{if } \alpha = 0,\ N \equiv 3 \pmod 4, \\ 4S(N), & \textit{if } \alpha = 1, \\ \left(2^{\alpha+1} - 2\left(\frac{2}{N}\right)\right) S(N), & \textit{if } \alpha \geq 2. \end{cases}$$

$$\text{(iv)} \quad N(1,2,2,8;n) = \begin{cases} 2S(N), & \textit{if } \alpha = 0,\ N \equiv 1 \pmod 4, \\ 4S(N), & \textit{if } \alpha = 0,\ N \equiv 3 \pmod 8, \\ 0, & \textit{if } \alpha = 0,\ N \equiv 7 \pmod 8, \\ 4S(N), & \textit{if } \alpha = 1, \\ \left(2^{\alpha+1} - 2\left(\frac{2}{N}\right)\right) S(N), & \textit{if } \alpha \geq 2. \end{cases}$$

$$\text{(v)} \quad N(1,2,4,4;n) = \begin{cases} 2S(N), & \textit{if } \alpha = 0, \\ \left(2^{\alpha+1} - 2\left(\frac{2}{N}\right)\right) S(N), & \textit{if } \alpha \geq 1. \end{cases}$$

$$\text{(vi)} \quad N(1,2,8,8;n) = \begin{cases} 2S(N), & \textit{if } \alpha = 0,\ N \equiv 1,3 \pmod 8, \\ 0, & \textit{if } \alpha = 0,\ N \equiv 5,7 \pmod 8, \\ 2S(N), & \textit{if } \alpha = 1, \\ \left(2^{\alpha} - 2\left(\frac{2}{N}\right)\right) S(N), & \textit{if } \alpha \geq 2. \end{cases}$$

We are now ready to determine $N(1,b,c,d;n)$ for the nineteen quaternary quadratic forms listed in (1.2). It is convenient to do this in the following order in order to exploit results from earlier theorems in the proofs of later theorems:

$x^2 + 2y^2 + 4z^2 + 8t^2$ Theorem 4.1

$x^2 + y^2 + 8z^2 + 8t^2$ Theorem 4.2

$x^2 + y^2 + 2z^2 + 8t^2$ Theorem 4.3

$x^2 + 4y^2 + 8z^2 + 8t^2$ Theorem 4.4

$x^2 + 4y^2 + 16z^2 + 16t^2$ Theorem 4.5

$x^2 + y^2 + 16z^2 + 16t^2$ Theorem 4.6

$x^2 + 4y^2 + 4z^2 + 16t^2$ Theorem 4.7

$x^2 + y^2 + 4z^2 + 16t^2$ Theorem 4.8

$x^2 + 2y^2 + 2z^2 + 16t^2$ Theorem 4.9

$x^2 + y^2 + z^2 + 16t^2$ Theorem 4.10

$x^2 + 8y^2 + 8z^2 + 16t^2$ Theorem 4.11

$x^2 + 16y^2 + 16z^2 + 16t^2$ Theorem 4.12

$x^2 + 2y^2 + 8z^2 + 16t^2$ Theorem 4.13

$x^2 + 4y^2 + 8z^2 + 16t^2$ Theorem 4.14

$x^2 + y^2 + 2z^2 + 16t^2$ Theorem 4.15

$x^2 + y^2 + 8z^2 + 16t^2$ Theorem 4.16

$x^2 + 2y^2 + 4z^2 + 16t^2$ Theorem 4.17

$x^2 + 2y^2 + 16z^2 + 16t^2$ Theorem 4.18

$x^2 + 8y^2 + 16z^2 + 16t^2$ Theorem 4.19

We give full details for the proofs of Theorems 4.1, 4.2, 4.10 and 4.19, and brief details for the remaining theorems, all of which can be proved similarly. All these results were claimed by Liouville in [7]- [25] without proof.

Theorem 4.1. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha N$, *where* $\alpha \in \mathbb{N}_0,\ N \in \mathbb{N}$ *and* $N \equiv 1 \pmod 2$. *Then*

$$N(1,2,4,8;n) = \begin{cases} \sigma(N) + \left(\dfrac{2}{N}\right) K_2(N), & \text{if } \alpha = 0,\ N \equiv 1 \pmod 4, \\ \sigma(N), & \text{if } \alpha = 0,\ N \equiv 3 \pmod 4, \\ 2\sigma(N), & \text{if } \alpha = 1, \\ 4\sigma(N), & \text{if } \alpha = 2, \\ 8\sigma(N), & \text{if } \alpha = 3, \\ 24\sigma(N), & \text{if } \alpha \geq 4. \end{cases}$$

Proof. We have by (4.5), Theorem 3.1(i), Theorem 2.4(i) and Theorem 2.3(iv)

$$\sum_{\substack{n=1 \\ n \equiv 1 \pmod 2}}^{\infty} N(1,2,4,8;n) q^n = \frac{1}{2}(\varphi(q) - \varphi(-q))\varphi(q^2)\varphi(q^4)\varphi(q^8)$$

$$= \frac{1}{4}(\varphi(q) - \varphi(-q))(\varphi^2(q^2)\varphi(q^4) + \varphi(-q^4)\varphi^2(-q^8))$$
$$= \sum_{\substack{n=1 \\ n\equiv 1 (\text{mod } 2)}}^{\infty} \sigma(n)q^n + \sum_{\substack{n=1 \\ n\equiv 1 (\text{mod } 2)}}^{\infty} \left(\frac{2}{n}\right) K_2(n)q^n.$$

Equating coefficients of q^n ($n \in \mathbb{N},\ n \equiv 1$ (mod 2)), we obtain

$$N(1,2,4,8;n) = \sigma(n) + \left(\frac{2}{n}\right) K_2(n),\ \text{if } n \equiv 1\,(\text{mod } 2),$$

which in view of (2.17) gives the assertion of the theorem when $\alpha = 0$.

When $n \equiv 0$ (mod 2), by Proposition 4.1(ix), we have

$$N(1,2,4,8;n) = N(1,2,2,4;n/2),\ \text{if } n \equiv 0\,(\text{mod } 2),$$

and the result now follows from Proposition 4.2(iv). □

Theorem 4.2. *Let* $n \in \mathbb{N}$*. Set* $n = 2^\alpha N$*, where* $\alpha \in \mathbb{N}_0,\ N \in \mathbb{N}$ *and* $N \equiv 1$ (mod 2)*. Then*

$$N(1,1,8,8;n) = \begin{cases} 2\sigma(N) + 2\left(\frac{2}{N}\right)K_2(N), & \text{if } \alpha = 0,\ N \equiv 1\,(\text{mod } 4), \\ 0, & \text{if } \alpha = 0,\ N \equiv 3\,(\text{mod } 4), \\ 4\sigma(N), & \text{if } \alpha = 1,\ N \equiv 1\,(\text{mod } 4), \\ 0, & \text{if } \alpha = 1,\ N \equiv 3\,(\text{mod } 4), \\ 4\sigma(N), & \text{if } \alpha = 2, \\ 8\sigma(N), & \text{if } \alpha = 3, \\ 24\sigma(N), & \text{if } \alpha \geq 4. \end{cases}$$

Proof. By Proposition 4.1(iv), (4.5), Theorem 3.1(ii), Theorem 2.4(ii) and Theorem 2.3(iv), we have

$$\sum_{\substack{n=1 \\ n\equiv 1 (\text{mod } 2)}}^{\infty} N(1,1,8,8;n)q^n$$
$$= 2 \sum_{\substack{n=1 \\ n\equiv 1 (\text{mod } 2)}}^{\infty} N(1,4,8,8;n)q^n$$
$$= (\varphi(q) - \varphi(-q))\varphi(q^4)\varphi^2(q^8)$$

$$
\begin{aligned}
&= \frac{1}{2}(\varphi(q) - \varphi(-q))\varphi^3(q^4) + \frac{1}{2}(\varphi(q) - \varphi(-q))\varphi(-q^4)\varphi^2(-q^8) \\
&= 2 \sum_{\substack{n=1 \\ n \equiv 1 (\mathrm{mod}\ 4)}}^{\infty} \sigma(n) q^n + 2 \sum_{n=1}^{\infty} \left(\frac{2}{n}\right) K_2(n) q^n.
\end{aligned}
$$

Equating coefficients of q^n $(n \in \mathbb{N},\ n \equiv 1$ (mod 2)), we deduce by (2.17),

$$
N(1,1,8,8;n) = \begin{cases} 2\sigma(N) + 2\left(\frac{2}{n}\right)K_2(n), & \text{if } n \equiv 1 \ (\mathrm{mod}\ 4), \\ 2\left(\frac{2}{n}\right)K_2(n) = 0, & \text{if } n \equiv 3 \ (\mathrm{mod}\ 4). \end{cases}
$$

If $n \equiv 0$ (mod 2) then

$$
N(1,1,8,8;n) = N(1,4,4;n/2),
$$

by Proposition 4.1(iii), and the result follows from Proposition 4.2(iii). □

Theorem 4.3. *Let* $n \in \mathbb{N}$*. Set* $n = 2^\alpha N$*, where* $\alpha \in \mathbb{N}_0,\ N \in \mathbb{N}$ *and* $N \equiv 1$ (mod 2)*. Then*

$$
N(1,1,2,8;n) = \begin{cases} 2\sigma(N) + 2\left(\frac{2}{N}\right)K_2(N), & \text{if } \alpha = 0,\ N \equiv 1 \ (\mathrm{mod}\ 4), \\ 2\sigma(N), & \text{if } \alpha = 0,\ N \equiv 3 \ (\mathrm{mod}\ 4), \\ 6\sigma(N), & \text{if } \alpha = 1,\ N \equiv 1 \ (\mathrm{mod}\ 4), \\ 2\sigma(N), & \text{if } \alpha = 1,\ N \equiv 3 \ (\mathrm{mod}\ 4), \\ 12\sigma(N), & \text{if } \alpha = 2, \\ 8\sigma(N), & \text{if } \alpha = 3, \\ 24\sigma(N), & \text{if } \alpha \geq 4. \end{cases}
$$

Proof. If $\alpha = 0$ by Proposition 4.1(iv) we have

$$
N(1,1,2,8;n) = 2N(1,2,4,8;n).
$$

Appealing to Theorem 4.1 we obtain the first two assertions of the theorem. If $\alpha \geq 1$, by Proposition 4.1(iii), we have

$$
N(1,1,2,8;n) = N(1,1,1,4;n/2),
$$

and the remaining assertions of the theorem follow from Proposition 4.2(i). □

Theorem 4.4. *Let* $n \in \mathbb{N}$. *Set* $n = 2^{\alpha}N$, *where* $\alpha \in \mathbb{N}_0,\ N \in \mathbb{N}$ *and* $N \equiv 1 \pmod 2$. *Then*

$$N(1,4,8,8;n) = \begin{cases} \sigma(N) + \left(\dfrac{2}{N}\right)K_2(N), & \text{if } \alpha = 0,\ N \equiv 1 \,(\mathrm{mod}\ 4), \\ 0, & \text{if } \alpha = 0,\ N \equiv 3 \,(\mathrm{mod}\ 4), \\ 0, & \text{if } \alpha = 1, \\ 4\sigma(N), & \text{if } \alpha = 2, \\ 8\sigma(N), & \text{if } \alpha = 3, \\ 24\sigma(N), & \text{if } \alpha \geq 4. \end{cases}$$

Proof. If $\alpha = 0$, by Proposition 4.1(iv), we have

$$N(1,1,8,8;n) = 2N(1,4,8,8;n),$$

and, by Theorem 4.2, we obtain the case $\alpha = 0$ of the theorem. If $\alpha = 1$, by Proposition 4.1(xi), we have $N(1,4,8,8;n) = 0$. If $\alpha \geq 2$, by Proposition 4.1(x), we have

$$N(1,4,8,8;n) = N(1,1,2,2;n/4),$$

and the rest of the theorem follows from Proposition 4.2(ii). □

Theorem 4.5. *Let* $n \in \mathbb{N}$. *Set* $n = 2^{\alpha}N$, *where* $\alpha \in \mathbb{N}_0,\ N \in \mathbb{N}$ *and* $N \equiv 1 \pmod 2$. *Then*

$$N(1,4,16,16;n) = \begin{cases} \dfrac{1}{2}\sigma(N) + \dfrac{1}{2}\left(2 + \left(\dfrac{2}{N}\right)\right)K_2(N), & \text{if } \alpha = 0,\ N \equiv 1 \,(\mathrm{mod}\ 4), \\ 0, & \text{if } \alpha = 0,\ N \equiv 3 \,(\mathrm{mod}\ 4), \\ 0, & \text{if } \alpha = 1, \\ 4\sigma(N), & \text{if } \alpha = 2,\ N \equiv 1 \,(\mathrm{mod}\ 4), \\ 0, & \text{if } \alpha = 2,\ N \equiv 3 \,(\mathrm{mod}\ 4), \\ 4\sigma(N), & \text{if } \alpha = 3, \\ 8\sigma(N), & \text{if } \alpha = 4, \\ 24\sigma(N), & \text{if } \alpha \geq 5. \end{cases}$$

Proof. The case $\alpha = 0,\ N \equiv 1 \pmod 4$ follows using (4.6), Theorem 3.1(iii), Theorem 2.4(ii) and Theorem 2.3(i)(iv). The cases $\alpha = 0,\ N \equiv 3 \pmod 4$ and $\alpha = 1$ follow from Proposition 4.1(xi). The case $\alpha \geq 2$ follows from Proposition 4.1(x) and Proposition 4.2(iii). □

Theorem 4.6. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha N$, *where* $\alpha \in \mathbb{N}_0,\ N \in \mathbb{N}$ *and* $N \equiv 1 \pmod 2$. *Then*

$$N(1,1,16,16;n) = \begin{cases} \sigma(N) + \left(2 + \left(\frac{2}{N}\right)\right) K_2(N), & \text{if } \alpha = 0,\ N \equiv 1 \pmod 4, \\ 0, & \text{if } \alpha = 0,\ N \equiv 3 \pmod 4, \\ 2\sigma(N) + 2\left(\frac{2}{N}\right) K_2(N), & \text{if } \alpha = 1,\ N \equiv 1 \pmod 4, \\ 0, & \text{if } \alpha = 1,\ N \equiv 3 \pmod 4, \\ 4\sigma(N), & \text{if } \alpha = 2,\ N \equiv 1 \pmod 4, \\ 0, & \text{if } \alpha = 2,\ N \equiv 3 \pmod 4, \\ 4\sigma(N), & \text{if } \alpha = 3, \\ 8\sigma(N), & \text{if } \alpha = 4, \\ 24\sigma(N), & \text{if } \alpha \geq 5. \end{cases}$$

Proof. By Proposition 4.1(iv) we have

$$N(1,1,16,16;n) = 2N(1,4,16,16;n), \text{ if } n \equiv 1 \pmod 2.$$

Appealing to Theorem 4.5 we obtain the case $\alpha = 0$ of the theorem.

If $n \equiv 2 \pmod 4$ then, by Proposition 4.1(iii) and Theorem 4.2, we obtain the case $\alpha = 1$ of the theorem.

If $n \equiv 0 \pmod 4$ then, by Proposition 4.1(iii) and Theorem 4.2(iii), we obtain the case $\alpha \geq 2$ of the theorem. ☐

Theorem 4.7. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha N$, *where* $\alpha \in \mathbb{N}_0,\ N \in \mathbb{N}$ *and* $N \equiv 1 \pmod 2$. *Then*

$$N(1,4,4,16;n) = \begin{cases} \sigma(N) + K_2(N), & \text{if } \alpha = 0,\ N \equiv 1 \pmod 4, \\ 0, & \text{if } \alpha = 0,\ N \equiv 3 \pmod 4, \\ 0, & \text{if } \alpha = 1, \\ 6\sigma(N), & \text{if } \alpha = 2,\ N \equiv 1 \pmod 4, \\ 2\sigma(N), & \text{if } \alpha = 2,\ N \equiv 3 \pmod 4, \\ 12\sigma(N), & \text{if } \alpha = 3, \\ 8\sigma(N), & \text{if } \alpha = 4, \\ 24\sigma(N), & \text{if } \alpha \geq 5. \end{cases}$$

Proof. The case $\alpha = 0$ of the theorem follows on using (4.5), Theorem 3.1(iv), Theorem 2.4(ii) and Theorem 2.3(i). The case $\alpha = 1$ follows from Proposition 4.1(xi). The case $\alpha \geq 2$ follows from Proposition 4.1(x) and Proposition 4.2(i). □

Theorem 4.8. *Let $n \in \mathbb{N}$. Set $n = 2^\alpha N$, where $\alpha \in \mathbb{N}_0,\ N \in \mathbb{N}$ and $N \equiv 1 \pmod 2$. Then*

$$N(1,1,4,16;n) = \begin{cases} 2\sigma(N) + 2K_2(N), & \text{if } \alpha = 0,\ N \equiv 1 \pmod 4, \\ 0, & \text{if } \alpha = 0,\ N \equiv 3 \pmod 4, \\ 2\sigma(N) + 2\left(\dfrac{2}{N}\right)K_2(N), & \text{if } \alpha = 1,\ N \equiv 1 \pmod 4, \\ 2\sigma(N), & \text{if } \alpha = 1,\ N \equiv 3 \pmod 4, \\ 6\sigma(N), & \text{if } \alpha = 2,\ N \equiv 1 \pmod 4, \\ 2\sigma(N), & \text{if } \alpha = 2,\ N \equiv 3 \pmod 4, \\ 12\sigma(N), & \text{if } \alpha = 3, \\ 8\sigma(N), & \text{if } \alpha = 4, \\ 24\sigma(N), & \text{if } \alpha \geq 5. \end{cases}$$

Proof. If $n \equiv 1 \pmod 2$ we have by Proposition 4.1(iv)

$$N(1,1,4,16;n) = 2N(1,4,4,16;n)$$

and the case $\alpha = 0$ of the theorem follows from Theorem 4.7.

If $n \equiv 0 \pmod 2$ then by Proposition 4.1(iii) we have

$$N(1,1,4,16;n) = N(1,1,2,8;n/2)$$

and the case $\alpha \geq 1$ of the theorem follows from Theorem 4.3. □

Theorem 4.9. *Let $n \in \mathbb{N}$. Set $n = 2^\alpha N$, where $\alpha \in \mathbb{N}_0,\ N \in \mathbb{N}$ and $N \equiv 1 \pmod 2$.*

Then

$$N(1,2,2,16;n)=\begin{cases}\sigma(N)+K_2(N), & \text{if } \alpha=0,\ N\equiv 1\ (\mathrm{mod}\ 4),\\ 2\sigma(N), & \text{if } \alpha=0,\ N\equiv 3\ (\mathrm{mod}\ 8),\\ 0, & \text{if } \alpha=0,\ N\equiv 7\ (\mathrm{mod}\ 8),\\ 2\sigma(N)+2\left(\dfrac{2}{N}\right)K_2(N), & \text{if } \alpha=1,\ N\equiv 1\ (\mathrm{mod}\ 4),\\ 2\sigma(N), & \text{if } \alpha=1,\ N\equiv 3\ (\mathrm{mod}\ 4),\\ 6\sigma(N), & \text{if } \alpha=2,\ N\equiv 1\ (\mathrm{mod}\ 4),\\ 2\sigma(N), & \text{if } \alpha=2,\ N\equiv 3\ (\mathrm{mod}\ 4),\\ 12\sigma(N), & \text{if } \alpha=3,\\ 8\sigma(N), & \text{if } \alpha=4,\\ 24\sigma(N), & \text{if } \alpha\geq 5.\end{cases}$$

Proof. The case $\alpha = 0$ of the theorem follows from (4.5), Theorem 3.1(v), Theorem 2.4(ii), Theorem 2.3(i) and Theorem 2.4(v). The case $\alpha \geq 1$ follows from Proposition 4.1(vii), (4.3) and Theorem 4.3. □

Theorem 4.10. *Let* $n \in \mathbb{N}$. *Set* $n = 2^{\alpha}N$, *where* $\alpha \in \mathbb{N}_0,\ N \in \mathbb{N}$ *and* $N \equiv 1$ (mod 2). *Then*

$$N(1,1,1,16;n)=\begin{cases}3\sigma(N)+3K_2(N), & \text{if } \alpha=0,\ N\equiv 1\ (\mathrm{mod}\ 4),\\ 2\sigma(N), & \text{if } \alpha=0,\ N\equiv 3\ (\mathrm{mod}\ 8),\\ 0, & \text{if } \alpha=0,\ N\equiv 7\ (\mathrm{mod}\ 8),\\ 6\sigma(N)+6\left(\dfrac{2}{N}\right)K_2(N), & \text{if } \alpha=1,\ N\equiv 1\ (\mathrm{mod}\ 4),\\ 6\sigma(N), & \text{if } \alpha=1,\ N\equiv 3\ (\mathrm{mod}\ 4),\\ 6\sigma(N), & \text{if } \alpha=2,\ N\equiv 1\ (\mathrm{mod}\ 4),\\ 2\sigma(N), & \text{if } \alpha=2,\ N\equiv 3\ (\mathrm{mod}\ 4),\\ 12\sigma(N), & \text{if } \alpha=3,\\ 8\sigma(N), & \text{if } \alpha=4,\\ 24\sigma(N), & \text{if } \alpha\geq 5.\end{cases}$$

Proof. Appealing to (4.4), Theorem 3.1(vi), Theorem 2.4(ii)(v) and Theorem 2.3(i), we obtain

$$\begin{aligned}
&\sum_{\substack{n=0\\ n\equiv 1\ (\mathrm{mod}\ 2)}}^{\infty} N(1,1,1,16;n)q^n \\
&= \frac{1}{2}\sum_{n=0}^{\infty} N(1,1,1,16;n)q^n - \frac{1}{2}\sum_{n=0}^{\infty} N(1,1,1,16;n)(-q)^n \\
&= \frac{1}{2}\varphi^3(q)\varphi(q^{16}) - \frac{1}{2}\varphi^3(-q)\varphi(q^{16}) \\
&= \frac{1}{2}(\varphi^3(q) - \varphi^3(-q))\varphi(q^{16}) \\
&= \frac{1}{2}(\varphi(q) - \varphi(-q))(\varphi^2(q) + \varphi(q)\varphi(-q) + \varphi^2(-q))\varphi(q^{16}) \\
&= \frac{1}{2}(\varphi(q) - \varphi(-q))\Big(\frac{3}{2}\varphi^3(q^4) + (\varphi(q^2) - \varphi(-q^2))\varphi(q^8)\varphi(q^{16}) \\
&\qquad\qquad + \frac{3}{2}\varphi(q^4)\varphi^2(-q^8)\Big) \\
&= \frac{3}{4}(\varphi(q) - \varphi(-q))\varphi^3(q^4) \\
&\quad + \frac{1}{2}(\varphi(q) - \varphi(-q))(\varphi(q^2) - \varphi(-q^2))\varphi(q^8)\varphi(q^{16}) \\
&\quad + \frac{3}{4}(\varphi(q) - \varphi(-q))\varphi(q^4)\varphi^2(-q^8) \\
&= 3\sum_{\substack{n=1\\ n\equiv 1\ (\mathrm{mod}\ 4)}}^{\infty} \sigma(n)q^n + 2\sum_{\substack{n=1\\ n\equiv 3\ (\mathrm{mod}\ 8)}}^{\infty} \sigma(n)q^n + 3\sum_{\substack{n=1\\ n\equiv 1\ (\mathrm{mod}\ 4)}}^{\infty} K_2(n)q^n.
\end{aligned}$$

Equating coefficients of q^n $(n \equiv 1 \pmod 2)$, we obtain

$$N(1,1,1,16;n) = \begin{cases} 3\sigma(n) + 3K_2(n), & \text{if } n \equiv 1 \pmod 4, \\ 2\sigma(N), & \text{if } n \equiv 3 \pmod 8, \\ 0, & \text{if } n \equiv 7 \pmod 8, \end{cases}$$

completing the case $\alpha = 0$.

The case $\alpha = 1$ follows from Proposition 4.1(ii) and Theorem 4.3.

The case $\alpha \geq 2$ follows from Proposition 4.1(i) and Proposition 4.2(i). □

Theorem 4.11. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha N$, *where* $\alpha \in \mathbb{N}_0,\ N \in \mathbb{N}$ *and* $N \equiv 1 \pmod 2$. *Then*

$$N(1,8,8,16;n) = \begin{cases} \sigma(N) + K_2(N), & \text{if } \alpha = 0,\ N \equiv 1 \,(\mathrm{mod}\ 8), \\ 0, & \text{if } \alpha = 0,\ N \equiv 3,5,7 \,(\mathrm{mod}\ 8), \\ 0, & \text{if } \alpha = 1, \\ 2\sigma(N), & \text{if } \alpha = 2, \\ 4\sigma(N), & \text{if } \alpha = 3, \\ 8\sigma(N), & \text{if } \alpha = 4, \\ 24\sigma(N), & \text{if } \alpha \geq 5. \end{cases}$$

Proof. The case $\alpha = 0,\ N \equiv 1$ (mod 8) follows from (4.7), Theorem 3.1(vii), Theorem 2.4(iv) and Theorem 2.3(ii). The cases $\alpha = 0,\ N \not\equiv 1$ (mod 8) and $\alpha = 1$ follow from Proposition 4.1(xii). The case $\alpha \geq 2$ follows from Proposition 4.1(x) and Proposition 4.2(iv). □

Theorem 4.12. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha N$, *where* $\alpha \in \mathbb{N}_0,\ N \in \mathbb{N}$ *and* $N \equiv 1 \pmod 2$. *Then*

$$N(1,16,16,16;n) = \begin{cases} \frac{1}{2}\sigma(N) + \frac{3}{2}K_2(N), & \text{if } \alpha = 0,\ N \equiv 1 \,(\mathrm{mod}\ 8), \\ 0, & \text{if } \alpha = 0,\ N \equiv 3,5,7 \,(\mathrm{mod}\ 8), \\ 0, & \text{if } \alpha = 1, \\ 2\sigma(N), & \text{if } \alpha = 2,\ N \equiv 1 \,(\mathrm{mod}\ 4), \\ 0, & \text{if } \alpha = 2,\ N \equiv 3 \,(\mathrm{mod}\ 4), \\ 0, & \text{if } \alpha = 3, \\ 8\sigma(N), & \text{if } \alpha = 4, \\ 24\sigma(N), & \text{if } \alpha \geq 5. \end{cases}$$

Proof. The case $\alpha = 0,\ N \equiv 1$ (mod 8) follows from (4.7), Theorem 3.1(viii), Theorem 2.4(iv) and Theorem 2.3(ii). The cases $\alpha = 0,\ N \not\equiv 1$ (mod 8) and $\alpha = 1$ follow from Proposition 4.1(xii). The case $\alpha \geq 2$ follows from Proposition 4.1(x) and Proposition 4.2(v). □

Theorem 4.13. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha N$, *where* $\alpha \in \mathbb{N}_0,\ N \in \mathbb{N}$ *and* $N \equiv 1 \pmod 2$.

Then

$$N(1,2,8,16;n) = \begin{cases} \sigma(N)+K_2(N), & \text{if } \alpha=0,\ N\equiv 1 \pmod 8, \\ \sigma(N), & \text{if } \alpha=0,\ N\equiv 3 \pmod 8, \\ 0, & \text{if } \alpha=0,\ N\equiv 5,7 \pmod 8, \\ \sigma(N)+\left(\frac{2}{N}\right)K_2(N), & \text{if } \alpha=1,\ N\equiv 1 \pmod 4, \\ \sigma(N), & \text{if } \alpha=1,\ N\equiv 3 \pmod 4, \\ 2\sigma(N), & \text{if } \alpha=2, \\ 4\sigma(N), & \text{if } \alpha=3, \\ 8\sigma(N), & \text{if } \alpha=4, \\ 24\sigma(N), & \text{if } \alpha\geq 5. \end{cases}$$

Proof. The case $\alpha = 0$ follows from (4.5), Theorem 3.1(ix), Theorem 2.4(iv)(v) and Theorem 2.3(ii). The case $\alpha \geq 1$ follows from Proposition 4.1(vii) and Theorem 4.1. □

Theorem 4.14. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha N$, *where* $\alpha \in \mathbb{N}_0,\ N \in \mathbb{N}$ *and* $N \equiv 1 \pmod 2$. *Then*

$$N(1,4,8,16;n) = \begin{cases} S(N)+K_1(N), & \text{if } \alpha=0,\ N\equiv 1 \pmod 8, \\ S(N), & \text{if } \alpha=0,\ N\equiv 5 \pmod 8, \\ 0, & \text{if } \alpha=0,\ N\equiv 3 \pmod 4, \\ 0, & \text{if } \alpha=1, \\ 4S(N), & \text{if } \alpha=2, \\ \left(2^\alpha - 2\left(\frac{2}{N}\right)\right)S(N), & \text{if } \alpha\geq 3. \end{cases}$$

Proof. The case $\alpha = 0$ follows from (4.5), Theorem 3.1(i), (2.8), Theorem 2.5(iii) and Theorem 2.2(ii). The case $\alpha = 1$ follows from Proposition 4.1(xi). The case $\alpha \geq 2$ follows from Proposition 4.1(x) and Proposition 4.3(ii). □

Theorem 4.15. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha N$, *where* $\alpha \in \mathbb{N}_0,\ N \in \mathbb{N}$ *and* $N \equiv 1 \pmod 2$.

Then

$$N(1,1,2,16;n) = \begin{cases} 2S(N) + 2K_1(N), & \text{if } \alpha = 0, \\ 6S(N), & \text{if } \alpha = 1,\ N \equiv 1 \pmod 4, \\ 4S(N), & \text{if } \alpha = 1,\ N \equiv 3 \pmod 8, \\ 0, & \text{if } \alpha = 1,\ N \equiv 7 \pmod 8, \\ 12S(N), & \text{if } \alpha = 2, \\ \left(2^\alpha - 2\left(\frac{2}{N}\right)\right)S(N), & \text{if } \alpha \geq 3. \end{cases}$$

Proof. The case $\alpha = 0$ follows from Proposition 4.1(iv), (4.3), (4.5), (2.6), (2.8), Theorem 2.5(i) and Theorem 2.2(i). The case $\alpha \geq 1$ follows from Proposition 4.1(iii) and Proposition 4.3(i). □

Theorem 4.16. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha N$, *where* $\alpha \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $N \equiv 1 \pmod 2$. *Then*

$$N(1,1,8,16;n) = \begin{cases} 2S(N) + 2K_1(N), & \text{if } \alpha = 0,\ N \equiv 1 \pmod 8, \\ 2S(N), & \text{if } \alpha = 0,\ N \equiv 5 \pmod 8, \\ 0, & \text{if } \alpha = 0,\ N \equiv 3 \pmod 4, \\ 4S(N), & \text{if } \alpha = 1,\ N \equiv 1 \pmod 4, \\ 0, & \text{if } \alpha = 1,\ N \equiv 3 \pmod 4, \\ 4S(N), & \text{if } \alpha = 2, \\ \left(2^\alpha - 2\left(\frac{2}{N}\right)\right)S(N), & \text{if } \alpha \geq 3. \end{cases}$$

Proof. By Proposition 4.1(iv) we have

$$N(1,1,8,16;n) = 2N(1,4,8,16;n), \text{ if } n \equiv 1 \pmod 2.$$

Appealing to Theorem 4.14 we obtain the case $\alpha = 0$ of the theorem.

By Proposition 4.1(iii) we have

$$N(1,1,8,16;n) = 2N(1,1,4,8;n/2), \text{ if } n \equiv 0 \pmod 2.$$

The case $\alpha \geq 1$ now follows from Proposition 4.3(iii). □

Theorem 4.17. *Let* $n \in \mathbb{N}$. *Set* $n = 2^{\alpha}N$, *where* $\alpha \in \mathbb{N}_0,\ N \in \mathbb{N}$ *and* $N \equiv 1 \pmod 2$. *Then*

$$N(1,2,4,16;n) = \begin{cases} S(N) + K_1(N), & \text{if } \alpha = 0,\ N \equiv 1,3 \pmod 8, \\ S(N), & \text{if } \alpha = 0,\ N \equiv 5,7 \pmod 8, \\ 2S(N), & \text{if } \alpha = 1,\ N \equiv 1 \pmod 4, \\ 4S(N), & \text{if } \alpha = 1,\ N \equiv 3 \pmod 8, \\ 0, & \text{if } \alpha = 1,\ N \equiv 7 \pmod 8, \\ 4S(N), & \text{if } \alpha = 2, \\ \left(2^{\alpha} - 2\left(\dfrac{2}{N}\right)\right)S(N), & \text{if } \alpha \geq 3. \end{cases}$$

Proof. The case $\alpha = 0$ follows from (4.5), (2.6), (2.8), Theorem 2.5(i) and Theorem 2.2(i). The case $\alpha \geq 1$ follows from Proposition 4.1(vii) and Proposition 4.3(iv). □

Theorem 4.18. *Let* $n \in \mathbb{N}$. *Set* $n = 2^{\alpha}N$, *where* $\alpha \in \mathbb{N}_0,\ N \in \mathbb{N}$ *and* $N \equiv 1 \pmod 2$. *Then*

$$N(1,2,16,16;n) = \begin{cases} S(N) + K_1(N), & \text{if } \alpha = 0,\ N \equiv 1,3 \pmod 8, \\ 0, & \text{if } \alpha = 0,\ N \equiv 5,7 \pmod 8, \\ 2S(N), & \text{if } \alpha = 1,\ N \equiv 1,3 \pmod 8, \\ 0, & \text{if } \alpha = 1,\ N \equiv 5,7 \pmod 8, \\ 2S(N), & \text{if } \alpha = 2, \\ \left(2^{\alpha-1} - 2\left(\dfrac{2}{N}\right)\right)S(N), & \text{if } \alpha \geq 3. \end{cases}$$

Proof. The case $\alpha = 0$ follows from (4.5), Theorem 3.1(x), Theorem 2.5(v)(vi) and Theorem 2.2(ii)(iii). The case $\alpha \geq 1$ follows from Proposition 4.1(vii) and Proposition 4.3(vi). □

Theorem 4.19. *Let* $n \in \mathbb{N}$. *Set* $n = 2^{\alpha}N$, *where* $\alpha \in \mathbb{N}_0,\ N \in \mathbb{N}$ *and* $N \equiv 1 \pmod 2$.

Then

$$N(1,8,16,16;n) = \begin{cases} S(N) + K_1(N), & \text{if } \alpha = 0,\ N \equiv 1 \pmod 8, \\ 0, & \text{if } \alpha = 0,\ N \equiv 3,5,7 \pmod 8, \\ 0, & \text{if } \alpha = 1, \\ 2S(N), & \text{if } \alpha = 2, \\ \left(2^{\alpha-1} - 2\left(\frac{2}{N}\right)\right) S(N), & \text{if } \alpha \geq 3. \end{cases}$$

Proof. By (4.7), (2.7), Theorem 2.5(v) and Theorem 2.2(ii), we obtain

$$\begin{aligned} \sum_{\substack{n=1 \\ n \equiv 1 \pmod 8}}^{\infty} & N(1,8,16,16;n) q^n \\ &= \frac{1}{2}(\varphi(q) - \varphi(-q))\varphi(q^8)\varphi^2(q^{16}) \\ &= \frac{1}{4}(\varphi(q) - \varphi(-q))\varphi(q^8)(\varphi^2(q^8) + \varphi^2(-q^8)) \\ &= \frac{1}{4}(\varphi(q) - \varphi(-q))\varphi^3(q^8) + \frac{1}{4}(\varphi(q) - \varphi(-q))\varphi(q^8)\varphi^2(-q^8) \\ &= \sum_{\substack{n=1 \\ n \equiv 1 \pmod 8}}^{\infty} S(n) q^n + \sum_{\substack{n=1 \\ n \equiv 1 \pmod 8}}^{\infty} K_1(n) q^n. \end{aligned}$$

Equating coefficients of q^n ($n \equiv 1 \pmod 8$), we obtain the case $\alpha = 0,\ N \equiv 1 \pmod 8$ of the theorem.

By Proposition 4.1(xii) we have

$$N(1,8,16,16;n) = 0, \text{ if } n \equiv 2,3,5,6,7 \pmod 8,$$

giving the cases $\alpha = 0,\ N \equiv 3,5,7 \pmod 8$ and $\alpha = 1$ of the theorem.

By Proposition 4.1(x) we have

$$N(1,8,16,16;n) = N(1,2,4,4;n/4), \text{ if } n \equiv 0 \pmod 4,$$

and the case $\alpha \geq 2$ of the theorem now follows from Proposition 4.3(v). □

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