International Mathematical Forum, 2, 2007, no. 2, 45 - 68

Evaluation of the convolution sums $\sum_{l+18m=n} \sigma(l)\sigma(m)$ and $\sum_{2l+9m=n} \sigma(l)\sigma(m)$

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Abstract

The convolution sums $\sum_{l+18m=n} \sigma(l)\sigma(m)$ and $\sum_{2l+9m=n} \sigma(l)\sigma(m)$ are evaluated for all positive integers n. These evaluations are used to determine the number of representations of a positive integer n by the forms $x_1^2+x_1x_2+x_2^2+x_3^2+x_3x_4+x_4^2+6(x_5^2+x_5x_6+x_6^2+x_7^2+x_7x_8+x_8^2)$ and $2(x_1^2+x_1x_2+x_2^2+x_3^2+x_3x_4+x_4^2)+3(x_5^2+x_5x_6+x_6^2+x_7^2+x_7x_8+x_8^2)$.

Mathematics Subject Classification: 11A25, 11E20, 11E25.

Keywords: sum of divisors function, convolution sums, Eisenstein series.

1 Introduction

Let $\mathbb{N}$ denote the set of natural numbers. For $k, n \in \mathbb{N}$ we set

$$\sigma_k(n) := \sum_{d \mid n} d^k,$$

where d runs through the positive divisors of n. If $n \notin \mathbb{N}$ we set $\sigma_k(n) = 0$. We also set $\sigma(n) := \sigma_1(n)$. In a recent paper [13] the third author showed that

$$\sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+9m=n}} \sigma(l)\sigma(m) = \frac{1}{216}\sigma_3(n) + \frac{1}{27}\sigma_3\left(\frac{n}{3}\right) + \frac{3}{8}\sigma_3\left(\frac{n}{9}\right) + \left(\frac{1}{24} - \frac{n}{36}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma\left(\frac{n}{9}\right) - \frac{1}{54}c_{1,9}(n), \quad (1.1)$$

where the integers $c_{1,9}(n)$ $(n \in \mathbb{N})$ are given by

$$\sum_{n=1}^{\infty} c_{1,9}(n)q^n := q\prod_{n=1}^{\infty}\left(1-q^{3n}\right)^8. \quad (1.2)$$

Clearly $c_{1,9}(1) = 1$ and

$$c_{1,9}(n) = 0, \quad \text{if } n \equiv 0, 2 \pmod 3. \quad (1.3)$$

In this paper we evaluate the analogous convolution sums

$$\sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+18m=n}} \sigma(l)\sigma(m) \quad \text{and} \quad \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ 2l+9m=n}} \sigma(l)\sigma(m),$$

see Theorem 2.1. For other convolution sums, see [1], [2], [4], [5], [6], [9], [13] and [14].

Let $\mathbb{Z}$ denote the set of all integers. For $k, l, n \in \mathbb{N}$ with $k \leq l$ we let

$$N_{k,l}(n) := \text{card}\{(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \in \mathbb{Z}^8 \mid n = k(x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2) + l(x_5^2 + x_5x_6 + x_6^2 + x_7^2 + x_7x_8 + x_8^2)\}. \quad (1.4)$$

We set $N_k(n) := N_{1,k}(n)$. The values of $N_k(n)$ are known for $k = 1, 2, 3, 4$ and 8, see [10], [4], [13], [1] and [2] respectively. In Section 4 we use the evaluations of $\sum_{l+18m=n} \sigma(l)\sigma(m)$ and $\sum_{2l+9m=n} \sigma(l)\sigma(m)$ to determine $N_6(n)$ and $N_{2,3}(n)$ for all $n \in \mathbb{N}$, see Theorem 2.2.

2 Statements of Theorems 2.1 and 2.2

We define quantities $c_{1,18}(n)$ and $c_{2,9}(n)$ for $n \in \mathbb{N}$.

Definition 2.1 *For $n \in \mathbb{N}$ we define $c_{1,18}(n)$ by*

$$
\begin{aligned}
31 \sum_{n=1}^{\infty} & c_{1,18}(n) q^n \\
&= 4q \prod_{n=1}^{\infty} (1-q^n)^2 (1-q^{2n})^2 (1-q^{3n})^2 (1-q^{6n})^2 \\
&+13q \prod_{n=1}^{\infty} (1+q^{2n})^2 (1-q^{2n-1})^4 (1+q^{3n})^4 (1-q^{6n})^8 (1-q^{12n-6})^6 \\
&+14q \prod_{n=1}^{\infty} (1+q^{2n})^2 (1-q^{2n-1}) (1+q^{3n})^3 (1-q^{6n})^8 (1-q^{12n-6})^6 \\
&+64q^2 \prod_{n=1}^{\infty} (1-q^{2n-1})^3 (1+q^{3n}) (1-q^{6n})^8 \\
&+68q^2 \prod_{n=1}^{\infty} (1-q^{6n})^8 \\
&+52q^3 \prod_{n=1}^{\infty} (1-q^n)^2 (1-q^{3n})^2 (1+q^{4n}+q^{8n})^2 (1-q^{12n})^4 \\
&+56q^3 \prod_{n=1}^{\infty} (1+q^n) (1-q^{3n})^3 (1-q^{4n-2})^2 (1+q^{6n}) (1-q^{12n})^5
\end{aligned}
$$

and $c_{2,9}(n)$ by

$$
\begin{aligned}
\sum_{n=1}^{\infty} & c_{2,9}(n) q^n \\
&= 3q \prod_{n=1}^{\infty} (1+q^{2n})^2 (1-q^{2n-1})^4 (1+q^{3n})^4 (1-q^{6n})^8 (1-q^{12n-6})^6 \\
&+4q \prod_{n=1}^{\infty} (1-q^n)^2 (1-q^{2n})^2 (1-q^{3n})^2 (1-q^{6n})^2
\end{aligned}
$$

$$-6q\prod_{n=1}^{\infty}(1+q^{2n})^2(1-q^{2n-1})(1+q^{3n})^3(1-q^{6n})^8(1-q^{12n-6})^6$$
$$-16q^2\prod_{n=1}^{\infty}(1-q^{2n-1})^3(1+q^{3n})(1-q^{6n})^8$$
$$+28q^2\prod_{n=1}^{\infty}(1-q^{6n})^8$$
$$+12q^3\prod_{n=1}^{\infty}(1-q^{n})^2(1-q^{3n})^2(1+q^{4n}+q^{8n})^2(1-q^{12n})^4$$
$$-24q^3\prod_{n=1}^{\infty}(1+q^{n})(1-q^{3n})^3(1-q^{4n-2})^2(1+q^{6n})(1-q^{12n})^5.$$

It is clear from Definition 2.1 that

$$31c_{1,18}(n)\in\mathbb{Z},\quad c_{2,9}(n)\in\mathbb{Z}\quad(n\in\mathbb{N}). \tag{2.1}$$

A short table of values of $31c_{1,18}(n)$ and $c_{2,9}(n)$ follows.

n	$31c_{1,18}(n)$	$c_{2,9}(n)$	n	$31c_{1,18}(n)$	$c_{2,9}(n)$
1	31	1	16	976	496
2	58	−2	17	1134	−2646
3	36	36	18	216	216
4	4	−116	19	980	380
5	−54	126	20	−216	504
6	−72	−72	21	−576	−576
7	−136	344	22	−504	216
8	−248	−488	23	−1512	3528
9	−108	−108	24	−288	−288
10	−252	108	25	−3119	−449
11	−108	252	26	−2116	−4396
12	144	144	27	324	324
13	98	−1042	28	−2944	−1024
14	512	1472	29	−270	630
15	216	216	30	−432	−432

We note that

$$c_{1,18}(1) = c_{2,9}(1) = 1. \tag{2.2}$$

The table suggests that

$$31c_{1,18}(n) = c_{2,9}(n), \text{ if } n \equiv 0 \pmod 3. \tag{2.3}$$

We prove this and similar results in Section 5.

In Section 3 we prove the following theorem.

Theorem 2.1 *Let $n \in \mathbb{N}$. Then*

(a)
$$\sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+18m=n}} \sigma(l)\sigma(m) = \frac{1}{1080}\sigma_3(n) + \frac{1}{270}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{135}\sigma_3\left(\frac{n}{3}\right) + \frac{4}{135}\sigma_3\left(\frac{n}{6}\right) + \frac{3}{40}\sigma_3\left(\frac{n}{9}\right) + \frac{3}{10}\sigma_3\left(\frac{n}{18}\right) + \left(\frac{1}{24} - \frac{n}{72}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma\left(\frac{n}{18}\right) - \frac{31}{1080}c_{1,18}(n)$$

and

(b)
$$\sum_{\substack{(l,m) \in \mathbb{N}^2 \\ 2l+9m=n}} \sigma(l)\sigma(m) = \frac{1}{1080}\sigma_3(n) + \frac{1}{270}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{135}\sigma_3\left(\frac{n}{3}\right) + \frac{4}{135}\sigma_3\left(\frac{n}{6}\right) + \frac{3}{40}\sigma_3\left(\frac{n}{9}\right) + \frac{3}{10}\sigma_3\left(\frac{n}{18}\right) + \left(\frac{1}{24} - \frac{n}{36}\right)\sigma\left(\frac{n}{2}\right) + \left(\frac{1}{24} - \frac{n}{8}\right)\sigma\left(\frac{n}{9}\right) - \frac{1}{1080}c_{2,9}(n).$$

In Section 4 we use Theorem 2.1 to prove the following result.

Theorem 2.2 *Let $n \in \mathbb{N}$. Then*

$$\begin{aligned}
\text{(a)}\quad N_6(n) &= \frac{4}{5}\sigma_3(n) + \frac{16}{5}\sigma_3\left(\frac{n}{2}\right) - \frac{88}{5}\sigma_3\left(\frac{n}{3}\right) \\
&\quad - \frac{352}{5}\sigma_3\left(\frac{n}{6}\right) + \frac{324}{5}\sigma_3\left(\frac{n}{9}\right) + \frac{1296}{5}\sigma_3\left(\frac{n}{18}\right) \\
&\quad + \frac{62}{5}c_{1,18}(n) - \frac{6}{5}c_{1,6}(n) - \frac{54}{5}c_{1,6}\left(\frac{n}{3}\right)
\end{aligned}$$

and

$$\begin{aligned}
\text{(b)}\quad N_{2,3}(n) &= \frac{4}{5}\sigma_3(n) + \frac{16}{5}\sigma_3\left(\frac{n}{2}\right) - \frac{88}{5}\sigma_3\left(\frac{n}{3}\right) \\
&\quad - \frac{352}{5}\sigma_3\left(\frac{n}{6}\right) + \frac{324}{5}\sigma_3\left(\frac{n}{9}\right) + \frac{1296}{5}\sigma_3\left(\frac{n}{18}\right) \\
&\quad + \frac{2}{5}c_{2,9}(n) - \frac{6}{5}c_{1,6}(n) - \frac{54}{5}c_{1,6}\left(\frac{n}{3}\right),
\end{aligned}$$

where $c_{1,6}(n)$ is given by

$$\sum_{n=1}^{\infty} c_{1,6}(n)q^n := q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{2n})^2(1-q^{3n})^2(1-q^{6n})^2. \quad (2.4)$$

3 Proof of Theorem 2.1

The Eisenstein series $L(q)$, $M(q)$ and $N(q)$ are defined by

$$L(q) := 1 - 24\sum_{n=1}^{\infty}\sigma(n)q^n, \ q \in \mathbb{C}, \ |q| < 1, \quad (3.1)$$

$$M(q) := 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n, \ q \in \mathbb{C}, \ |q| < 1, \quad (3.2)$$

$$N(q) := 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n, \ q \in \mathbb{C}, \ |q| < 1, \quad (3.3)$$

see for example [11, eqn. (25)], [12, p. 140]. Ramanujan's discriminant function $\Delta(q)$ is given by

$$\Delta(q) := q\prod_{n=1}^{\infty}(1-q^n)^{24} = \frac{1}{1728}\left(M(q)^3 - N(q)^2\right), \tag{3.4}$$

see for example [11, eqn. (44)], [12, p. 144]. The Jacobi theta function $\varphi(q)$ is defined by

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}, \quad q \in \mathbb{C}, \quad |q| < 1. \tag{3.5}$$

Set

$$p = p(q) := \frac{\varphi^2(q) - \varphi^2(q^3)}{2\varphi^2(q^3)} \tag{3.6}$$

and

$$k = k(q) := \frac{\varphi^3(q^3)}{\varphi(q)}. \tag{3.7}$$

The following result is proved in [3, Theorem 10].

Triplication principle.

$$\begin{aligned} p(q^3) = 3^{-1}\big((&-4-3p+6p^2+4p^3) \\ &+2^{2/3}(1-2p-2p^2)((1-p)(1+2p)(2+p))^{1/3} \\ &+ 2^{1/3}(1+2p)((1-p)(1+2p)(2+p))^{2/3}\big), \\ k(q^3) = 3^{-2}\big(&3+2^{2/3}(1+2p)((1-p)(1+2p)(2+p))^{1/3} \\ &+ 2^{4/3}((1-p)(1+2p)(2+p))^{2/3}\big)\,k. \end{aligned}$$

From [4, eqn. (3.87)] we have

$$L(q) - 3L(q^3) = -(2+16p+36p^2+16p^3+2p^4)k^2 \tag{3.8}$$

and from [4, eqn. (3.85)]

$$L(q^3) - 2L(q^6) = -(1+2p+2p^3+p^4)k^2. \tag{3.9}$$

Set

$$E := E(q) = 2^{-1/3}((1-p)(1+2p)(2+p))^{1/3}. \tag{3.10}$$

Applying the triplication principle to (3.8), we obtain

$$\begin{aligned} L(q^3) - 3L(q^9) = &-\frac{2}{9}(1+8p+18p^2+8p^3+p^4)k^2 \\ &-\frac{8}{9}(1+3p-3p^2-p^3)Ek^2 \\ &-\frac{8}{9}(1-2p+p^2)E^2k^2. \end{aligned} \tag{3.11}$$

Applying the triplication principle to (3.9), we deduce

$$\begin{aligned} L(q^9) - 2L(q^{18}) = &\frac{1}{27}(-11-10p+24p^2-10p^3-11p^4)k^2 \\ &-\frac{8}{27}(1-p^3)Ek^2 - \frac{8}{27}(1+4p+p^2)E^2k^2. \end{aligned} \tag{3.12}$$

Then, from (3.8), (3.11), (3.12) and the trivial identity

$$L(q) - 18L(q^{18}) = L(q) - 3L(q^3) + 3(L(q^3) - 3L(q^9)) + 9(L(q^9) - 2L(q^{18})),$$

we deduce

$$\begin{aligned} L(q) - 18L(q^{18}) = &-\frac{1}{3}(19+74p+120p^2+74p^3+19p^4)k^2 \\ &-\frac{8}{3}(2+3p-3p^2-2p^3)Ek^2 \\ &-\frac{16}{3}(1+p+p^2)E^2k^2. \end{aligned} \tag{3.13}$$

From [4, eqn. (3.84)] we have

$$L(q) - 2L(q^2) = -(1+14p+24p^2+14p^3+p^4)k^2. \tag{3.14}$$

Subtracting (3.14) from (3.8), we deduce

$$2L(q^2) - 3L(q^3) = -(1+2p+12p^2+2p^3+p^4)k^2. \tag{3.15}$$

Then, from (3.11), (3.15) and the trivial identity

$$2L(q^2) - 9L(q^9) = 2L(q^2) - 3L(q^3) + 3(L(q^3) - 3L(q^9))$$

we obtain

$$\begin{aligned} 2L(q^2) - 9L(q^9) = &-\frac{1}{3}(5 + 22p + 72p^2 + 22p^3 + 5p^4)k^2 \\ &-\frac{8}{3}(1 + 3p - 3p^2 - p^3)Ek^2 \\ &-\frac{8}{3}(1 - 2p + p^2)E^2k^2. \end{aligned} \tag{3.16}$$

Squaring (3.13) and (3.16) we have

Lemma 3.1

(a)
$$\begin{aligned} (L(q) - 18L(q^{18}))^2 = &(97 + 540p + 1300p^2 + 2044p^3 + 2442p^4 \\ &+2044p^5 + 1300p^6 + 540p^7 + 97p^8)k^4 \\ &+(96 + 464p + 848p^2 + 512p^3 - 512p^4 \\ &-848p^5 - 464p^6 - 96p^7)Ek^4 \\ &+(96 + 416p + 736p^2 + 768p^3 + 736p^4 \\ &+416p^5 + 96p^6)E^2k^4. \end{aligned}$$

(b)
$$\begin{aligned} (2L(q^2) - 9L(q^9))^2 = &(17 + 60p + 20p^2 + 284p^3 + 1002p^4 \\ &+284p^5 + 20p^6 + 60p^7 + 17p^8)k^4 \\ &+(16 + 48p + 208p^2 + 368p^3 - 368p^4 \\ &-208p^5 - 48p^6 - 16p^7)Ek^4 \\ &+(16 + 64p + 80p^2 - 320p^3 + 80p^4 \\ &+64p^5 + 16p^6)E^2k^4. \end{aligned}$$

From [4, eqns. (3.69), (3.70), (3.71) and (3.72)] we have

$$M(q) = (1 + 124p + 964p^2 + 2788p^3 + 3910p^4 \tag{3.17}$$

$$+2788p^5 + 964p^6 + 124p^7 + p^8)k^4.$$

$$M(q^2) = (1 + 4p + 64p^2 + 178p^3 + 235p^4 + 178p^5 + 64p^6 + 4p^7 + p^8)k^4, \tag{3.18}$$

$$M(q^3) = (1 + 4p + 4p^2 + 28p^3 + 70p^4 + 28p^5 + 4p^6 + 4p^7 + p^8)k^4, \tag{3.19}$$

$$M(q^6) = (1 + 4p + 4p^2 - 2p^3 - 5p^4 - 2p^5 + 4p^6 + 4p^7 + p^8)k^4. \tag{3.20}$$

Applying the triplication principle to (3.19) and (3.20), we obtain

$$\begin{aligned} M(q^9) = {} & \frac{1}{243}(83 + 212p - 628p^2 - 436p^3 + 1970p^4 \\ & -436p^5 - 628p^6 + 212p^7 + 83p^8)k^4 \\ & +\frac{1}{243}(80 + 160p + 960p^2 + 1520p^3 - 1520p^4 \\ & -960p^5 - 160p^6 - 80p^7)Ek^4 \\ & +\frac{1}{243}(80 + 480p + 240p^2 - 1600p^3 + 240p^4 \\ & +480p^5 + 80p^6)E^2k^4 \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} M(q^{18}) = {} & \frac{1}{243}\Big(83 + 332p + 272p^2 - 346p^3 - 655p^4 \\ & -346p^5 + 272p^6 + 332p^7 + 83p^8\Big)k^4 \\ & +\frac{1}{243}(80 + 280p + 360p^2 + 200p^3 - 200p^4 \\ & -360p^5 - 280p^6 - 80p^7)Ek^4. \end{aligned} \tag{3.22}$$

$$+\frac{1}{243}(80+240p+120p^2-160p^3+120p^4$$
$$+240p^5+80p^6)E^2k^4.$$

From (3.17)-(3.22) we deduce

Lemma 3.2

(a) $23M(q)-8M(q^2)-16M(q^3)-64M(q^6)$
$-162M(q^9)+7452M(q^{18})$

$$=(2425+12540p+30100p^2+52060p^3+65850p^4$$
$$+52060p^5+30100p^6+12540p^7+2425p^8)k^4$$
$$+(2400+8480p+10400p^2+5120p^3-5120p^4$$
$$-10400p^5-8480p^6-2400p^7)Ek^4$$
$$+(2400+7040p+3520p^2-3840p^3+3520p^4$$
$$+7040p^5+2400p^6)E^2k^4.$$

(b) $-2M(q)+92M(q^2)-16M(q^3)-64M(q^6)$
$+1863M(q^9)-648M(q^{18})$

$$=(425+540p-1900p^2+8060p^3+29850p^4$$
$$+8060p^5-1900p^6+540p^7+425p^8)k^4$$
$$+(400+480p+6400p^2+11120p^3-11120p^4$$
$$-6400p^5-480p^6-400p^7)Ek^4$$
$$+(400+3040p+1520p^2-11840p^3+1520p^4$$
$$+3040p^5+400p^6)E^2k^4.$$

From Lemmas 3.1 and 3.2 we obtain

Lemma 3.3

(a)
$$\begin{aligned}
&\left(L(q)-18L(q^{18})\right)^2\\
&\quad -\frac{1}{25}\Big(23M(q)-8M(q^2)-16M(q^3)-64M(q^6)\\
&\quad -162M(q^9)+7452M(q^{18})\Big)\\
&= \frac{96}{5}p(1-p)^2(1+p)^2(1+2p)(2+p)k^4\\
&\quad +\frac{24}{5}2^{2/3}p(1-p)(1+p)^2(13+32p+13p^2)((1-p)(1+2p)(2+p))^{1/3}k^4\\
&\quad +\frac{48}{5}2^{1/3}p(1+p)^2(7+17p+7p^2)((1-p)(1+2p)(2+p))^{2/3}k^4.
\end{aligned}$$

(b)
$$\begin{aligned}
&\left(2L(q^2)-9L(q^9)\right)^2\\
&\quad -\frac{1}{25}\Big(-2M(q)+92M(q^2)-16M(q^3)-64M(q^6)\\
&\quad +1863M(q^9)-648M(q^{18})\Big)\\
&= \frac{96}{5}p(1-p)^2(1+p)^2(1+2p)(2+p)k^4.\\
&\quad +\frac{24}{5}2^{2/3}p(1-p)(1+p)^2(3-8p+3p^2)((1-p)(1+2p)(2+p))^{1/3}k^4\\
&\quad -\frac{48}{5}2^{1/3}p(1+p)^2(3-7p+3p^2)((1-p)(1+2p)(2+p))^{2/3}k^4.
\end{aligned}$$

Before proceeding to the next lemma, we recall from [1, eqns. (3.28)-(3.33)]

$$\Delta(q)=\frac{1}{16}p(1+p)^4(1-p)^{12}(1+2p)^3(2+p)^3k^{12}, \tag{3.23}$$

$$\Delta(q^2)=\frac{1}{256}p^2(1+p)^2(1-p)^6(1+2p)^6(2+p)^6k^{12}, \tag{3.24}$$

$$\Delta(q^3)=\frac{1}{16}p^3(1+p)^{12}(1-p)^4(1+2p)(2+p)k^{12}, \tag{3.25}$$

$$\Delta(q^4) = \frac{1}{65536} p^4(1+p)(1-p)^3(1+2p)^3(2+p)^{12}k^{12}, \qquad (3.26)$$

$$\Delta(q^6) = \frac{1}{256} p^6(1+p)^6(1-p)^2(1+2p)^2(2+p)^2k^{12}, \qquad (3.27)$$

$$\Delta(q^{12}) = \frac{1}{65536} p^{12}(1+p)^3(1-p)(1+2p)(2+p)^4k^{12}. \qquad (3.28)$$

Lemma 3.4

(a) $$31\sum_{n=1}^{\infty} c_{1,18}(n)q^n$$
$$= p(1-p)^2(1+p)^2(1+2p)(2+p)k^4$$
$$+\frac{1}{4}2^{2/3}p(1-p)(1+p)^2(13+32p+13p^2)((1-p)(1+2p)(2+p))^{1/3}k^4$$
$$+\frac{1}{2}2^{1/3}p(1+p)^2(7+17p+7p^2)((1-p)(1+2p)(2+p))^{2/3}k^4.$$

(b) $$\sum_{n=1}^{\infty} c_{2,9}(n)q^n$$
$$= p(1-p)^2(1+p)^2(1+2p)(2+p)k^4$$
$$+\frac{1}{4}2^{2/3}p(1-p)(1+p)^2(3-8p+3p^2)((1-p)(1+2p)(2+p))^{1/3}k^4$$
$$-\frac{1}{2}2^{1/3}p(1+p)^2(3-7p+3p^2)((1-p)(1+2p)(2+p))^{2/3}k^4.$$

Proof. We just prove part (a) as part (b) can be proved similarly. We have

$$q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{2n})^2(1-q^{3n})^2(1-q^{6n})^2$$
$$= q\Big(\frac{\Delta(q)}{q}\Big)^{1/12}\Big(\frac{\Delta(q^2)}{q^2}\Big)^{1/12}\Big(\frac{\Delta(q^3)}{q^3}\Big)^{1/12}\Big(\frac{\Delta(q^6)}{q^6}\Big)^{1/12}$$
$$= \Delta(q)^{1/12}\Delta(q^2)^{1/12}\Delta(q^3)^{1/12}\Delta(q^6)^{1/12}$$
$$= \frac{1}{4}p(1+p)^2(1-p)^2(1+2p)(2+p)k^4,$$

$$\begin{aligned}
q\prod_{n=1}^{\infty}&(1+q^{2n})^2(1-q^{2n-1})^4(1+q^{3n})^4(1-q^{6n})^8(1-q^{12n-6})^6\\
&=q\prod_{n=1}^{\infty}(1-q^n)^4(1-q^{2n})^{-6}(1-q^{3n})^{-4}(1-q^{4n})^2(1-q^{6n})^{18}(1-q^{12n})^{-6}\\
&=q\Big(\frac{\Delta(q)}{q}\Big)^{1/6}\Big(\frac{\Delta(q^2)}{q^2}\Big)^{-1/4}\Big(\frac{\Delta(q^3)}{q^3}\Big)^{-1/6}\Big(\frac{\Delta(q^4)}{q^4}\Big)^{1/12}\\
&\qquad\times\Big(\frac{\Delta(q^6)}{q^6}\Big)^{3/4}\Big(\frac{\Delta(q^{12})}{q^{12}}\Big)^{-1/4}\\
&=\Delta(q)^{1/6}\Delta(q^2)^{-1/4}\Delta(q^3)^{-1/6}\Delta(q^4)^{1/12}\Delta(q^6)^{3/4}\Delta(q^{12})^{-1/4}\\
&=2^{-4/3}p(1+p)^2(1-p)^{4/3}(1+2p)^{1/3}(2+p)^{1/3}k^4,
\end{aligned}$$

$$\begin{aligned}
q\prod_{n=1}^{\infty}&(1+q^{2n})^2(1-q^{2n-1})(1+q^{3n})^3(1-q^{6n})^8(1-q^{12n-6})^6\\
&=q\prod_{n=1}^{\infty}(1-q^n)(1-q^{2n})^{-3}(1-q^{3n})^{-3}(1-q^{4n})^2(1-q^{6n})^{17}(1-q^{12n})^{-6}\\
&=q\Big(\frac{\Delta(q)}{q}\Big)^{1/24}\Big(\frac{\Delta(q^2)}{q^2}\Big)^{-1/8}\Big(\frac{\Delta(q^3)}{q^3}\Big)^{-1/8}\Big(\frac{\Delta(q^4)}{q^4}\Big)^{1/12}\\
&\qquad\times\Big(\frac{\Delta(q^6)}{q^6}\Big)^{17/24}\Big(\frac{\Delta(q^{12})}{q^{12}}\Big)^{-1/4}\\
&=\Delta(q)^{1/24}\Delta(q^2)^{-1/8}\Delta(q^3)^{-1/8}\Delta(q^4)^{1/12}\Delta(q^6)^{17/24}\Delta(q^{12})^{-1/4}\\
&=2^{-5/3}p(1+p)^2(1-p)^{2/3}(1+2p)^{2/3}(2+p)^{2/3}k^4,
\end{aligned}$$

$$\begin{aligned}
q^2\prod_{n=1}^{\infty}&(1-q^{2n-1})^3(1+q^{3n})(1-q^{6n})^8\\
&=q^2\prod_{n=1}^{\infty}(1-q^n)^3(1-q^{2n})^{-3}(1-q^{3n})^{-1}(1-q^{6n})^9\\
&=q^2\Big(\frac{\Delta(q)}{q}\Big)^{1/8}\Big(\frac{\Delta(q^2)}{q^2}\Big)^{-1/8}\Big(\frac{\Delta(q^3)}{q^3}\Big)^{-1/24}\Big(\frac{\Delta(q^6)}{q^6}\Big)^{3/8}
\end{aligned}$$

$$= \Delta(q)^{1/8}\Delta(q^2)^{-1/8}\Delta(q^3)^{-1/24}\Delta(q^6)^{3/8}$$
$$= 2^{-7/3}p^2(1+p)^2(1-p)^{4/3}(1+2p)^{1/3}(2+p)^{1/3}k^4,$$

$$q^2\prod_{n=1}^{\infty}(1-q^{6n})^8 = q^2\Big(\frac{\Delta(q^6)}{q^6}\Big)^{1/3} = \Delta(q^6)^{1/3}$$
$$= \frac{2^{1/3}}{8}p^2(1+p)^2(1-p)^{2/3}(1+2p)^{2/3}(2+p)^{2/3}k^4,$$

$$q^3\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{3n})^2(1+q^{4n}+q^{8n})^2(1-q^{12n})^4$$
$$= q^3\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{3n})^2(1-q^{4n})^{-2}(1-q^{12n})^6$$
$$= q^3\Big(\frac{\Delta(q)}{q}\Big)^{1/12}\Big(\frac{\Delta(q^3)}{q^3}\Big)^{1/12}\Big(\frac{\Delta(q^4)}{q^4}\Big)^{-1/12}\Big(\frac{\Delta(q^{12})}{q^{12}}\Big)^{1/4}$$
$$= \Delta(q)^{1/12}\Delta(q^3)^{1/12}\Delta(q^4)^{-1/12}\Delta(q^{12})^{1/4}$$
$$= \frac{2^{2/3}}{16}p^3(1+p)^2(1-p)^{4/3}(1+2p)^{1/3}(2+p)^{1/3}k^4,$$

$$q^3\prod_{n=1}^{\infty}(1+q^n)(1-q^{3n})^3(1-q^{4n-2})^2(1+q^{6n})(1-q^{12n})^5$$
$$= q^3\prod_{n=1}^{\infty}(1-q^n)^{-1}(1-q^{2n})^3(1-q^{3n})^3(1-q^{4n})^{-2}(1-q^{6n})^{-1}(1-q^{12n})^6$$
$$= q^3\Big(\frac{\Delta(q)}{q}\Big)^{-1/24}\Big(\frac{\Delta(q^2)}{q^2}\Big)^{1/8}\Big(\frac{\Delta(q^3)}{q^3}\Big)^{1/8}\Big(\frac{\Delta(q^4)}{q^4}\Big)^{-1/12}$$
$$\times\Big(\frac{\Delta(q^6)}{q^6}\Big)^{-1/24}\Big(\frac{\Delta(q^{12})}{q^{12}}\Big)^{1/4}$$
$$= \Delta(q)^{-1/24}\Delta(q^2)^{1/8}\Delta(q^3)^{1/8}\Delta(q^4)^{-1/12}\Delta(q^6)^{-1/24}\Delta(q^{12})^{1/4}$$

$$= 2^{-11/3}p^3(1+p)^2(1-p)^{2/3}(1+2p)^{2/3}(2+p)^{2/3}k^4.$$

Thus, by Definition 2.1, we have

$$\begin{aligned}
&31\sum_{n=1}^{\infty} c_{1,18}(n)q^n \\
&= p(1+p)^2(1-p)^2(1+2p)(2+p)k^4 \\
&\quad +13\cdot 2^{-4/3}p(1+p)^2(1-p)^{4/3}(1+2p)^{1/3}(2+p)^{1/3}k^4 \\
&\quad +14\cdot 2^{-5/3}p(1+p)^2(1-p)^{2/3}(1+2p)^{2/3}(2+p)^{2/3}k^4 \\
&\quad +64\cdot 2^{-7/3}p^2(1+p)^2(1-p)^{4/3}(1+2p)^{1/3}(2+p)^{1/3}k^4 \\
&\quad +\frac{17}{2}2^{1/3}p^2(1+p)^2(1-p)^{2/3}(1+2p)^{2/3}(2+p)^{2/3}k^4 \\
&\quad +\frac{13}{4}2^{2/3}p^3(1+p)^2(1-p)^{4/3}(1+2p)^{1/3}(2+p)^{1/3}k^4 \\
&\quad +56\cdot 2^{-11/3}p^3(1+p)^2(1-p)^{2/3}(1+2p)^{2/3}(2+p)^{2/3}k^4 \\
&= p(1+p)^2(1-p)^2(1+2p)(2+p)k^4 \\
&\quad +\frac{1}{4}2^{2/3}p(1+p)^2(1-p)(13+32p+13p^2)((1-p)(1+2p)(2+p))^{1/3}k^4 \\
&\quad +\frac{1}{2}2^{1/3}p(1+p)^2(7+17p+7p^2)((1-p)(1+2p)(2+p))^{2/3}k^4,
\end{aligned}$$

as asserted. ■

From Lemmas 3.3 and 3.4 we have

Lemma 3.5

(a) $\left(L(q)-18L(q^{18})\right)^2$

$$\begin{aligned}
&= \frac{23}{25}M(q)-\frac{8}{25}M(q^2)-\frac{16}{25}M(q^3)-\frac{64}{25}M(q^6) \\
&\quad -\frac{162}{25}M(q^9)+\frac{7452}{25}M(q^{18})+\frac{2976}{5}\sum_{n=1}^{\infty}c_{1,18}(n)q^n.
\end{aligned}$$

(b) $\left(2L(q^2)-9L(q^9)\right)^2$

$$= -\frac{2}{25}M(q) + \frac{276}{75}M(q^2) - \frac{48}{75}M(q^3) - \frac{192}{75}M(q^6) + \frac{5589}{75}M(q^9) - \frac{1944}{75}M(q^{18}) + \frac{96}{5}\sum_{n=1}^{\infty} c_{2,9}(n)q^n.$$

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. We just prove part (a) as part (b) can be treated similarly.

We begin by recalling the classical identity

$$L(q)^2 = 1 + \sum_{n=1}^{\infty}(240\sigma_3(n) - 288n\sigma(n))q^n, \tag{3.29}$$

see for example [7], [8]. Mapping $q \to q^{18}$ in (3.29), we obtain

$$L(q^{18})^2 = 1 + \sum_{n=1}^{\infty}\left(240\sigma_3\left(\frac{n}{18}\right) - 16n\sigma\left(\frac{n}{18}\right)\right)q^n. \tag{3.30}$$

Also

$$\begin{aligned} L(q)L(q^{18}) &= \left(1 - 24\sum_{n=1}^{\infty}\sigma(n)q^n\right)\left(1 - 24\sum_{n=1}^{\infty}\sigma(n)q^{18n}\right) \\ &= 1 - 24\sum_{n=1}^{\infty}\sigma(n)q^n - 24\sum_{n=1}^{\infty}\sigma\left(\frac{n}{18}\right)q^n \\ &\quad +576\sum_{n=1}^{\infty}W_{18}(n)q^n, \end{aligned} \tag{3.31}$$

where

$$W_{a,b}(n) := \sum_{\substack{(l,m)\in\mathbb{N}^2 \\ al+bm=n}} \sigma(l)\sigma(m),\ W_b(n) := W_{1,b}(n),\ a,b,n\in\mathbb{N}. \tag{3.32}$$

Thus, from (3.29)–(3.31), we have

$$\begin{aligned}
&\left(L(q)-18L(q^{18})\right)^2\\
&\quad = L(q)^2+324L(q^{18})^2-36L(q)L(q^{18})\\
&\quad = 289+\sum_{n=1}^{\infty}\Big(240\sigma_3(n)+77760\sigma_3\left(\frac{n}{18}\right)\\
&\qquad -288n\sigma(n)+864\sigma(n)-5184n\sigma\left(\frac{n}{18}\right)+864\sigma\left(\frac{n}{18}\right)\\
&\qquad -20736W_{18}(n)\Big)q^n.
\end{aligned}$$

From (3.2) and Lemma 3.4(a) we have

$$\begin{aligned}
&\left(L(q)-18L(q^{18})\right)^2\\
&\quad = 289+\sum_{n=1}^{\infty}\Big(\frac{1104}{5}\sigma_3(n)-\frac{384}{5}\sigma_3\left(\frac{n}{2}\right)-\frac{768}{5}\sigma_3\left(\frac{n}{3}\right)\\
&\qquad -\frac{3072}{5}\sigma_3\left(\frac{n}{6}\right)-\frac{7776}{5}\sigma_3\left(\frac{n}{9}\right)+\frac{357696}{5}\sigma_3\left(\frac{n}{18}\right)\\
&\qquad +\frac{2976}{5}c_{1,18}(n)\Big)q^n.
\end{aligned}$$

Equating the coefficients of q^n $(n\in\mathbb{N})$, we obtain

$$\begin{aligned}
&1104\sigma_3(n)-384\sigma_3\left(\frac{n}{2}\right)-768\sigma_3\left(\frac{n}{3}\right)-3072\sigma_3\left(\frac{n}{6}\right)\\
&\qquad -7776\sigma_3\left(\frac{n}{9}\right)+357696\sigma_3\left(\frac{n}{18}\right)+2976c_{1,18}(n)\\
&\quad = 1200\sigma_3(n)+388800\sigma_3\left(\frac{n}{18}\right)-1440n\sigma(n)+4320\sigma(n)\\
&\qquad -25920n\sigma\left(\frac{n}{18}\right)+4320\sigma\left(\frac{n}{18}\right)-103680W_{18}(n).
\end{aligned}$$

Solving for $W_{18}(n)$ we obtain the assertion of part (a) of Theorem 2.1. ■

4 Proof of Theorem 2.2

Let $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. For $l\in\mathbb{N}_0$ we set

$$r(l)=\operatorname{card}\{(x_1,x_2,x_3,x_4)\in\mathbb{Z}^4 \mid l=x_1^2+x_1x_2+x_2^2+x_3^2+x_3x_4+x_4^2\} \tag{4.1}$$

so that $r(0) = 1$. It is known that [9, Theorem 13], [10]

$$r(l) = 12\sigma(l) - 36\sigma\left(\frac{l}{3}\right), \quad l \in \mathbb{N}. \tag{4.2}$$

By (1.1) and (4.1) we have for $n \in \mathbb{N}$

$$\begin{aligned} N_6(n) &= \sum_{\substack{(l,m) \in \mathbb{N}_0^2 \\ l+6m=n}} r(l)r(m) \\ &= r(n)r(0) + r(0)r\left(\frac{n}{6}\right) + \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+6m=n}} r(l)r(m). \end{aligned}$$

Thus by (4.2) we obtain

$$\begin{aligned} N_6(n) &= 12\sigma(n) - 36\sigma\left(\frac{n}{3}\right) + 12\sigma\left(\frac{n}{6}\right) - 36\sigma\left(\frac{n}{18}\right) \\ &\quad +144 \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+6m=n}} \sigma(l)\sigma(m) - 432 \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+6m=n}} \sigma\left(\frac{l}{3}\right)\sigma(m) \\ &\quad -432 \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+6m=n}} \sigma(l)\sigma\left(\frac{m}{3}\right) + 1296 \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+6m=n}} \sigma\left(\frac{l}{3}\right)\sigma\left(\frac{m}{3}\right). \end{aligned} \tag{4.3}$$

By a result of Alaca and Williams [4, Theorem 1] the first sum is

$$\begin{aligned} \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+6m=n}} \sigma(l)\sigma(m) &= \frac{1}{120}\sigma_3(n) + \frac{1}{30}\sigma_3\left(\frac{n}{2}\right) + \frac{3}{40}\sigma_3\left(\frac{n}{3}\right) + \frac{3}{10}\sigma_3\left(\frac{n}{6}\right) \\ &\quad + \left(\frac{1}{24} - \frac{n}{24}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma\left(\frac{n}{6}\right) \\ &\quad - \frac{1}{120}c_{1,6}(n), \end{aligned}$$

where $c_{1,6}(n)$ ($n \in \mathbb{N}$) is given by (2.4).

By a result of Huard, Ou, Spearman and Williams [9, Theorem 2, p. 247] the second sum is

$$\sum_{\substack{(l,m)\in\mathbb{N}^2\\ l+6m=n}} \sigma\left(\frac{l}{3}\right)\sigma(m) = \sum_{\substack{(l,m)\in\mathbb{N}^2\\ l+2m=n/3}} \sigma(l)\sigma(m) = \sum_{\substack{m\in\mathbb{N}\\ m<n/6}} \sigma(m)\sigma\left(\frac{n}{3}-2m\right)$$

$$= \frac{1}{12}\sigma_3\left(\frac{n}{3}\right) + \frac{1}{3}\sigma_3\left(\frac{n}{6}\right) + \left(\frac{1}{24}-\frac{n}{24}\right)\sigma\left(\frac{n}{3}\right) + \left(\frac{1}{24}-\frac{n}{12}\right)\sigma\left(\frac{n}{6}\right).$$

By Theorem 2.1(a) the third sum is

$$\sum_{\substack{(l,m)\in\mathbb{N}^2\\ l+6m=n}} \sigma(l)\sigma\left(\frac{m}{3}\right) = \sum_{\substack{(l,m)\in\mathbb{N}^2\\ l+18m=n}} \sigma(l)\sigma(m)$$

$$= \frac{1}{1080}\sigma_3(n) + \frac{1}{270}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{135}\sigma_3\left(\frac{n}{3}\right) + \frac{4}{135}\sigma_3\left(\frac{n}{6}\right)$$

$$+\frac{3}{40}\sigma_3\left(\frac{n}{9}\right) + \frac{3}{10}\sigma_3\left(\frac{n}{18}\right) + \left(\frac{1}{24}-\frac{n}{72}\right)\sigma(n)$$

$$+\left(\frac{1}{24}-\frac{n}{4}\right)\sigma\left(\frac{n}{18}\right) - \frac{31}{1080}c_{1,18}(n).$$

Again by [4, Theorem 1] with n replaced by $n/3$, the fourth sum is

$$\sum_{\substack{(l,m)\in\mathbb{N}^2\\ l+6m=n}} \sigma\left(\frac{l}{3}\right)\sigma\left(\frac{m}{3}\right) = \sum_{\substack{(l,m)\in\mathbb{N}^2\\ l+6m=n/3}} \sigma(l)\sigma(m)$$

$$= \frac{1}{120}\sigma_3\left(\frac{n}{3}\right) + \frac{1}{30}\sigma_3\left(\frac{n}{6}\right) + \frac{3}{40}\sigma_3\left(\frac{n}{9}\right) + \frac{3}{10}\sigma_3\left(\frac{n}{18}\right)$$

$$+\left(\frac{1}{24}-\frac{n}{72}\right)\sigma\left(\frac{n}{3}\right) + \left(\frac{1}{24}-\frac{n}{12}\right)\sigma\left(\frac{n}{18}\right)$$

$$-\frac{1}{120}c_{1,6}\left(\frac{n}{3}\right).$$

Putting these evaluations into (4.3) we obtain part (a) of Theorem 2.2. Part (b) can be proved similarly. ∎

5 Proof of (2.3)

Let $N \in \mathbb{N}$. We have

$$W_{18}(3N) = \sum_{m<N/6} \sigma(m)\sigma(3(N-6m)).$$

Appealing to the simple identity

$$\sigma(3n) = 4\sigma(n) - 3\sigma\left(\frac{n}{3}\right),\ n \in \mathbb{N},$$

we obtain

$$W_{18}(3N) = 4W_6(N) - 3W_2\left(\frac{N}{3}\right). \tag{5.1}$$

Appealing to Theorem 2.1 for the value of $W_{18}(n)$, to [4, Theorem 1] for the value of $W_6(n)$, and to [9, Theorem 2] for the value of $W_2(n)$, we obtain

$$c_{1,18}(3N) = \frac{36}{31}c_{1,6}(N). \tag{5.2}$$

Similarly we have

$$W_{2,9}(3N) = 4W_{2,3}(N) - 3W_2\left(\frac{N}{3}\right) \tag{5.3}$$

and

$$c_{2,9}(3N) = 36c_{1,6}(N). \tag{5.4}$$

From (5.2) and (5.4) we obtain (2.3). ■

It was shown in [4] that

$$c_{1,6}(2^\alpha 3^\beta) = (-1)^{\alpha+\beta}2^\alpha 3^\beta,\ \alpha, \beta \in \mathbb{N}_0. \tag{5.5}$$

From (5.2), (5.4) and (5.5) we obtain

$$c_{1,18}(2^\alpha 3^\beta) = (-1)^{\alpha+\beta+1}2^{\alpha+2}3^{\beta+1}/31,\ \alpha \in \mathbb{N}_0\ ,\beta \in \mathbb{N}, \tag{5.6}$$

and

$$c_{2,9}(2^\alpha 3^\beta) = (-1)^{\alpha+\beta+1}2^{\alpha+2}3^{\beta+1},\ \alpha \in \mathbb{N}_0\ ,\beta \in \mathbb{N}. \tag{5.7}$$

In a manner similar to the proof of (2.3), we find

$$W_{1,18}(2N) + 2W_{2,9}(N) = 3W_9(N) \tag{5.8}$$

and

$$2W_{1,18}(N) + W_{2,9}(2N) = 3W_9(N). \tag{5.9}$$

Appealing to Theorem 2.1 and (1.1), we deduce that

$$31c_{1,18}(2N) + 2c_{2,9}(N) = 60c_{1,9}(N) \tag{5.10}$$

and

$$62c_{1,18}(N) + c_{2,9}(2N) = 60c_{1,9}(N). \tag{5.11}$$

Then, from (1.3), we have

$$31c_{1,18}(2N) = -2c_{2,9}(N), \text{ if } N \equiv 0, 2 \pmod 3, \tag{5.12}$$

and

$$62c_{1,18}(N) = -2c_{2,9}(2N), \text{ if } N \equiv 0, 2 \pmod 3. \tag{5.13}$$

Finally, appealing to (2.3), we obtain

$$c_{1,18}(2N) = -2c_{1,18}(N), \text{ if } N \equiv 0 \pmod 3, \tag{5.14}$$
$$c_{2,9}(2N) = -2c_{2,9}(N), \text{ if } N \equiv 0 \pmod 3. \tag{5.15}$$

ACKNOWLEDGEMENTS. Research of the third author was supported by Natural Sciences and Engineering Research Council of Canada grant A-7233.

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Received: May 10, 2006