CYCLIC QUARTIC FIELDS WITH A UNIQUE NORMAL INTEGRAL BASIS

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Abstract

Cyclic quartic fields possessing a unique normal integral basis are characterized and the normal integral basis is given explicitly.

1. Introduction

Let $K$ be an abelian extension of finite degree over the rational field $\mathbb{Q}$. By the Kronecker-Weber theorem [8, Theorem 6.5, p. 289], $K$ is contained in a cyclotomic field $\mathbb{Q}(\zeta_m) \quad (\zeta_m = e^{2\pi i / m})$ for some positive integer $m$. The least positive integer $f$ with the property $K \subseteq \mathbb{Q}(\zeta_f)$ is called the conductor of $K$ and is denoted by $f(K)$ [8, p. 421]. Moreover [8, Proposition 8.1, p. 421]

$$K \subseteq \mathbb{Q}(\zeta_m) \ 	ext{if and only if} \ f(K) \mid m. \quad (1.1)$$

It is known [8, p. 175] that an abelian field $K$ has a normal integral basis if and only if it is contained in a cyclotomic field $\mathbb{Q}(\zeta_m)$ with $m$.

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squarefree, that is, by (1.1) if and only if \( f(K) \) is squarefree. If \( K \) is abelian with squarefree conductor \( f(K) \), then the normal integral basis of \( K \) is unique (up to permutation and change of sign) precisely when either \( \text{Gal}(K/Q) \) is the direct product of cyclic groups of order 4 and/or order 2, or \( \text{Gal}(K/Q) \) is the direct product of cyclic groups of order 3 and/or order 2, see Thompson [10, Theorem 1, p. 1119] or Higman [4, Theorem 6].

In this note we determine explicitly a normal integral basis for \( K \) in radical form, when \( K \) is a cyclic quartic field with squarefree conductor. To do this we make use of the representation of \( K \) given by Hardy et al. [3, Theorem 1, p. 1], the formula for the conductor of \( K \) given in [3, Theorem 5, p. 34] or [9], and the integral basis for \( K \) given by Hudson and Williams [6, Theorem, p. 146]. From a theoretical point of view we can give the normal integral basis as the conjugates of \( T_{\mathbb{Q}(-\sqrt{f})/K}(\mathbb{Q}(\sqrt{f})) \) [8, Proposition 4.15(i), p. 174]. However this is not a practical way of obtaining a normal integral basis as the degree of \( \mathbb{Q}(\sqrt{f})/\mathbb{Q} \) can be large.

This was observed by Acciaro and Fieker [1] in the case of cyclic fields of prime degree.

It is shown in [3] that a cyclic quartic extension \( K/Q \) can be expressed uniquely in the form

\[
K = \mathbb{Q}(\sqrt[4]{A(D + B\sqrt{D})}),
\]

where \( A, B, C \) and \( D \) are integers such that

\[
A \text{ is squarefree and odd,}
\]

\[
D = B^2 + C^2 \text{ is squarefree, } B > 0, C > 0,
\]

\[
\text{GCD}(A, D) = 1.
\]

An algorithm to express \( K \) in this form has been given by Huard et al. [5]. The conductor of \( f(K) \) of \( K \) is given by

\[
f(K) = 2^t |A|D,
\]
where
\[
\begin{align*}
l &= 3, \quad \text{if } D \equiv 2 \pmod{4} \text{ or } D \equiv 1 \pmod{4}, \quad B \equiv 1 \pmod{2}, \\
&= 2, \quad \text{if } D \equiv 1 \pmod{4}, \quad B \equiv 0 \pmod{2}, \quad A + B \equiv 3 \pmod{4}, \\
&= 0, \quad \text{if } D \equiv 1 \pmod{4}, \quad B \equiv 0 \pmod{2}, \quad A + B \equiv 1 \pmod{4},
\end{align*}
\]  
(1.7)

see [3], [9]. Thus
\[
\begin{align*}
f(K) \text{ is squarefree } &\iff l = 0 \\
&\iff D \equiv 1 \pmod{4}, \quad B \equiv 0 \pmod{2}, \quad A + B \equiv 1 \pmod{4}.
\end{align*}
\]  
(1.8)

Now let \( K \) be a cyclic quartic extension of \( \mathbb{Q} \) with a squarefree conductor \( f(K) \) so that \( K \) has a unique normal integral basis. Then \( K \) can be expressed uniquely in the form (1.2), where \( A, B, C \) and \( D \) are integers satisfying (1.3)-(1.5) and
\[
D \equiv 1 \pmod{4}, \quad B \equiv 0 \pmod{2}, \quad C \equiv 1 \pmod{2}, \quad A + B \equiv 1 \pmod{4}.
\]  
(1.9)

We set
\[
\alpha = \sqrt{A(D + B \sqrt{D})}, \quad \beta = \sqrt{A(D - B \sqrt{D})},
\]  
(1.10)

and define \( \varepsilon = \pm 1 \) by
\[
A = \varepsilon C \pmod{4}.
\]  
(1.11)

An integral basis for \( K \) is
\[
\left\{ 1, \frac{1}{2} \left( 1 + \sqrt{D} \right), \frac{1}{4} \left( 1 + \sqrt{D} + \alpha + \varepsilon \beta \right), \frac{1}{4} \left( 1 - \sqrt{D} + \alpha - \varepsilon \beta \right) \right\},
\]  
(1.12)

see [6, Theorem, p. 146]. We prove

**Theorem.** Let \( K \) be a cyclic quartic field with a squarefree conductor. Then its unique (up to permutation and change of sign) normal integral basis consists of the following four elements

\[
\frac{1}{4} \left( 1 + \sqrt{D} + \alpha + \varepsilon \beta \right), \frac{1}{4} \left( 1 - \sqrt{D} - \alpha + \varepsilon \beta \right), \frac{1}{4} \left( 1 + \sqrt{D} - \alpha - \varepsilon \beta \right)
\]

and

\[
\frac{1}{4} \left( 1 - \sqrt{D} + \alpha - \varepsilon \beta \right),
\]

where \( \alpha \) and \( \beta \) are defined in (1.10) and \( \varepsilon \) in (1.11).
2. Proof of Theorem

The first element specified in the theorem, namely \( \frac{1}{4}(1 + \sqrt{D} + \alpha + \varepsilon \beta) \), is the third element in the integral basis given in (1.12) and so is an integer of \( K \). Using the automorphism of \( K \) given by

\[
\alpha \rightarrow \beta, \quad \beta \rightarrow -\alpha, \quad \sqrt{D} \rightarrow -\sqrt{D},
\]

we see that the other three elements given in the theorem are the conjugates of \( \frac{1}{4}(1 + \sqrt{D} + \alpha + \varepsilon \beta) \) and so are also integers of \( K \). Since the first and fourth elements in the theorem are members of (1.12), the sum of all four elements is 1, and the sum of the first and third elements is \( \frac{1}{2}(1 + \sqrt{D}) \), it is clear that the four elements of the theorem comprise a normal integral basis for \( K \).

3. Example

Let \( K = \mathbb{Q}(\sqrt{-(13 + 2\sqrt{13})}) \). Here

\[
A = -1, \quad B = 2, \quad C = 3, \quad D = 13, \quad \varepsilon = 1,
\]

\[
\alpha = \sqrt{-13 - 2\sqrt{13}}, \quad \beta = \sqrt{-13 + 2\sqrt{13}},
\]

so by the theorem a normal integral basis for \( K \) consists of the four elements

\[
\frac{1}{4}(1 + \sqrt{13} + \sqrt{-13 - 2\sqrt{13}} + \sqrt{-13 + 2\sqrt{13}}),
\]

\[
\frac{1}{4}(1 - \sqrt{13} - \sqrt{-13 - 2\sqrt{13}} + \sqrt{-13 + 2\sqrt{13}}),
\]

\[
\frac{1}{4}(1 + \sqrt{13} - \sqrt{-13 - 2\sqrt{13}} - \sqrt{-13 + 2\sqrt{13}}),
\]

\[
\frac{1}{4}(1 - \sqrt{13} + \sqrt{-13 - 2\sqrt{13}} - \sqrt{-13 + 2\sqrt{13}}).
\]

We conclude by determining the unique normal integral basis of \( K \) by
means of a trace calculation. By (1.6) and (1.7) the conductor of $K$ is $f(K) = 13$. Let $\omega = \zeta_{13} = e^{2\pi i/13}$. We have $[Q(\omega) : Q] = \phi(13) = 12. [K : Q] = 4$ and $[Q(\omega) : K] = 3$. We determine

$$Tr_{Q(\omega)/K}(\omega) = \omega + \omega^3 + \omega^9$$

and its conjugates. From the evaluation of the quartic Gauss sum given by Matthews [7], see [2, Theorem 4.2.4, p. 163] (with $p = 13$, $a = 3$, $|b| = 2$, $C = 1$), we have

$$1 + 4(\omega + \omega^3 + \omega^9) = \sum_{n=0}^{12} \omega^{n^4} = \sqrt{13} + i\sqrt{26 - 6\sqrt{13}}.$$  

Hence

$$\omega + \omega^3 + \omega^9 = \frac{1}{4} \left(-1 + \sqrt{13} + \sqrt{-26 + 6\sqrt{13}}\right)$$

$$= \frac{1}{4} \left(-1 + \sqrt{13} + \sqrt{-13 - 2\sqrt{13}} - \sqrt{-13 + 2\sqrt{13}}\right).$$

Similarly we have

$$\omega^2 + \omega^5 + \omega^6 = \frac{1}{4} \left(-1 - \sqrt{13} + \sqrt{-26 - 6\sqrt{13}}\right)$$

$$= \frac{1}{4} \left(-1 - \sqrt{13} + \sqrt{-13 - 2\sqrt{13}} + \sqrt{-13 + 2\sqrt{13}}\right),$$

$$\omega^4 + \omega^{10} + \omega^{12} = \frac{1}{4} \left(-1 + \sqrt{13} - \sqrt{-26 + 6\sqrt{13}}\right)$$

$$= \frac{1}{4} \left(-1 + \sqrt{13} - \sqrt{-13 - 2\sqrt{13}} + \sqrt{-13 + 2\sqrt{13}}\right),$$

$$\omega^7 + \omega^8 + \omega^{11} = \frac{1}{4} \left(-1 - \sqrt{13} + \sqrt{-26 - 6\sqrt{13}}\right)$$

$$= \frac{1}{4} \left(-1 - \sqrt{13} - \sqrt{-13 - 2\sqrt{13}} + \sqrt{-13 + 2\sqrt{13}}\right).$$

Thus the normal integral basis obtained from $Tr_{Q(\omega)/K}(\omega)$ and its conjugates is (as it should be!) the same as the one determined above.
References


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