ON REAL CUBIC FIELDS WITH A UNIQUE FUNDAMENTAL UNIT

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Abstract

Simple conditions are given which ensure that the cubic equation

\[ x^3 + tx^2 + ux - 1 = 0 \quad (t, u \in \mathbb{Z}) \]

has a unique real root \( \theta > 1 \) which is the unique fundamental unit \((\theta > 1)\) of the cubic field \( K = \mathbb{Q}(\theta) \).

1. Introduction

Let \( t \) and \( u \) be integers. Set

\[ D = 4t^3 + u^2t^2 - 18ut - (4u^3 + 27). \quad (1.1) \]

We suppose that

\[ t + u < 0, \quad (1.2) \]

\[ t - u \neq 2, \quad (1.3) \]

\[ D < 0. \quad (1.4) \]

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Conditions (1.2) and (1.3) ensure that the cubic polynomial

\[ f(x) = x^3 + tx^2 + ux - 1 \] (1.5)

satisfies

\[ f(1) < 0, \quad f(-1) \neq 0. \] (1.6)

Thus \( f(x) \) does not have \( \pm 1 \) as roots and thus is irreducible in \( \mathbb{Z}[x] \). The integer \( D \) is the discriminant of \( f(x) \) [4, p. 83]. In view of (1.4), \( f(x) \) has exactly one real root \( \theta \). Moreover, by the first inequality in (1.6) we have

\[ \theta > 1. \] (1.7)

Let \( r + is \) and \( r - is \) \((r \in \mathbb{R}, s \in \mathbb{R} \setminus \{0\})\) be the two nonreal roots of \( f(x) \). Then

\[ x^3 + tx^2 + ux - 1 = (x - \theta)(x - (r + is))(x - (r - is)) \]
\[ = x^3 - (\theta + 2r)x^2 + (2r\theta + (r^2 + s^2))x - \theta(r^2 + s^2). \]

Thus

\[ t = -\theta - 2r, \quad u = 2r\theta + (r^2 + s^2), \quad 1 = \theta(r^2 + s^2). \]

By (1.7) we have

\[ r^2 + s^2 = \frac{1}{\theta} < 1 \]

so that \(|r| < 1\). Hence

\[ t = -\theta - 2r < -1 - 2r \leq -1 + 2|r| < -1 + 2 = 1. \]

As \( t \in \mathbb{Z} \) we deduce that

\[ t \leq 0. \] (1.8)

Further we have

\[ t^2 - 2u = (\theta + 2r)^2 - 2(2r\theta + (r^2 + s^2)) = \theta^2 + 2r^2 - 2s^2 \]
\[ > 1 - 2(r^2 + s^2) > 1 - 2 = -1 \]
so that as \( t^2 - 2u \in \mathbb{Z} \) we deduce

\[
t^2 \geq 2u. \tag{1.9}
\]

Set \( K = \mathbb{Q}(\theta) \). The field \( K \) is a real cubic field whose two conjugate fields are nonreal. Thus, by Dirichlet's unit theorem [2, Theorem 3.6, p. 101], the ring \( O_K \) of integers of \( K \) has a unique fundamental unit \( \eta > 1 \) such that every unit of \( O_K \) is of the form \( \pm \eta^k \) for some \( k \in \mathbb{Z} \). As \( \theta \in K \) and \( \theta \) is a root of a monic integral polynomial (namely \( f(x) \)), we have \( \theta \in O_K \). As \( \theta(\theta^2 + t\theta + u) = 1 \), we see that \( \theta|1 \) in \( O_K \). Thus \( \theta \) is a unit of \( O_K \). Hence \( \theta = \pm \eta^k \) for some \( k \in \mathbb{Z} \). As \( \theta > 1 \) and \( \eta > 1 \) we must have

\[
\theta = \eta^k, \quad k \in \mathbb{N}. \tag{1.10}
\]

In this note we give a simple criterion on \( t \) and \( u \) which ensures that \( \theta = \eta \).

2. Criterion for Fundamental Unit \( \eta \) of \( O_K \) to be \( \theta \)

With the notation of Section 1, we prove the following theorem.

**Theorem.** Let \( M = M(t, u) \) be the largest positive integer such that

\[
M^2 | D, \quad \frac{D}{M^2} \equiv 0 \text{ or } 1 \pmod{4}, \quad \frac{|D|}{M^2} \geq 23. \tag{2.1}
\]

Let \( m = m(t, u) \) be a real number such that

\[
f(m) > 0, \quad m \geq \left( \frac{3}{2} \right)^{\frac{2}{3}} (= 1.31 \text{ approx}). \tag{2.2}
\]

If

\[
M \leq \frac{|D|^{1/2}}{(27 + 4m^{3/2})^{1/2}} \tag{2.3}
\]

then

\[
\theta = \eta. \tag{2.4}
\]
Proof. We denote the discriminant of $K$ by $d(K)$ and the index of $\theta$ by $\text{ind} \, \theta$ so that

$$D = (\text{ind} \, \theta)^2 d(K). \quad (2.5)$$

Thus $(\text{ind} \, \theta)^2 \mid D$. By Stickelberger's theorem [2, Theorem 2.6, p. 59] we have $d(K) \equiv 0$ or $1 \pmod{4}$ so that

$$\frac{D}{(\text{ind} \, \theta)^2} = 0 \text{ or } 1 \pmod{4}. \quad (2.6)$$

As $K$ is a real cubic field with two nonreal conjugate fields, we have $|d(K)| \geq 23$ so that

$$\frac{|D|}{(\text{ind} \, \theta)^2} \geq 23, \quad (2.7)$$

[3, Table 3.2, p. 437]. By the maximality of $M$ we deduce that

$$\text{ind} \, \theta \leq M. \quad (2.8)$$

Hence by (2.3) we have

$$|d(K)| = \frac{|D|}{(\text{ind} \, \theta)^2} \geq \frac{|D|}{M^2} \geq 27 + 4m^{3/2}. \quad (2.9)$$

Thus, as $m \geq \left(\frac{3}{2}\right)^2$, we have

$$|d(K)| \geq 27 + 4 \cdot \frac{3}{2} = 33. \quad (2.10)$$

Then, by [1, Question 35, pp. 152-153], we deduce that

$$\eta^3 > \frac{|d(K)| - 27}{4}. \quad (2.11)$$

From (2.9) and (2.11), we obtain

$$\eta^3 > m^{3/2} \quad (2.12)$$
and so

$$\eta^2 > m. \quad (2.13)$$

Since \( f \) has a unique real root \( \theta > 1 \) and \( f(m) > 0 \), we must have

$$m > \theta. \quad (2.14)$$

From (1.7), (2.13) and (2.14), we deduce that

$$1 < \theta < \eta^2. \quad (2.15)$$

Then, from (1.10) and (2.15), we obtain \( \theta = \eta \).

We emphasize that it is not necessary to know the discriminant \( d(K) \) of the cubic field \( K \) in order to apply the theorem.

3. Polynomials having Fundamental Unit \( \eta \) as a Root

Running through those integers \( t \) between \(-39\) and 0 (recall (1.8)) and those integers \( u \) between \(-39\) and 39, which satisfy (1.2), (1.3) and \(-1000 < D < 0\), we obtain the following table of polynomials

\( f(x) = x^3 + tx^2 + ux - 1 \) of discriminant \( D \) having the fundamental unit \( \eta \) as a root.

It should be noted that the theorem does not always find the fundamental unit of a real cubic field with two nonreal embeddings although it does so in a great many cases. For example if \( K = Q(\theta) \), where \( \theta \in \mathbb{R} \) satisfies \( \theta^3 - \theta^2 - 1 = 0 \), then it can be deduced from [3, Table 3.2, p. 437] that the fundamental unit (> 1) of \( O_K \) is \( \theta \). However in this case

\[
D = -31, \ M = 1, \ m = 1.466 \ (\text{approx}), \ \frac{1}{\left| D \right|^\frac{3}{2}} \left( \frac{3}{1} \right) = 0.953 \ (\text{approx}) < 1.
\]

\[
\frac{1}{27 + 4m^2}^2
\]
<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>$D$</th>
<th>$M$</th>
<th>$m$</th>
<th>$\frac{1}{\left(D \frac{1}{2}\right)^{3/2} + 4m^2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^3 - 1x^2 - 1x - 1$</td>
<td>-44</td>
<td>1</td>
<td>1.840</td>
<td>1.091</td>
</tr>
<tr>
<td>$x^3 - 2x^2 + 0x - 1$</td>
<td>-59</td>
<td>1</td>
<td>2.206</td>
<td>1.213</td>
</tr>
<tr>
<td>$x^3 - 3x^2 + 1x - 1$</td>
<td>-76</td>
<td>1</td>
<td>2.770</td>
<td>1.293</td>
</tr>
<tr>
<td>$x^3 - 2x^2 - 2x - 1$</td>
<td>-83</td>
<td>1</td>
<td>2.832</td>
<td>1.342</td>
</tr>
<tr>
<td>$x^3 - 2x^2 - 1x - 1$</td>
<td>-87</td>
<td>1</td>
<td>2.547</td>
<td>1.418</td>
</tr>
<tr>
<td>$x^3 - 4x^2 + 2x - 1$</td>
<td>-107</td>
<td>1</td>
<td>3.512</td>
<td>1.417</td>
</tr>
<tr>
<td>$x^3 - 3x^2 + 0x - 1$</td>
<td>-135</td>
<td>1</td>
<td>3.104</td>
<td>1.662</td>
</tr>
<tr>
<td>$x^3 - 6x^2 + 4x - 1$</td>
<td>-139</td>
<td>1</td>
<td>5.279</td>
<td>1.357</td>
</tr>
<tr>
<td>$x^3 - 3x^2 - 2x - 1$</td>
<td>-175</td>
<td>1</td>
<td>3.628</td>
<td>1.790</td>
</tr>
<tr>
<td>$x^3 - 4x^2 + 1x - 1$</td>
<td>-199</td>
<td>1</td>
<td>3.807</td>
<td>1.873</td>
</tr>
<tr>
<td>$x^3 - 10x^2 + 6x - 1$</td>
<td>-211</td>
<td>1</td>
<td>9.372</td>
<td>1.220</td>
</tr>
<tr>
<td>$x^3 - 5x^2 - 4x - 1$</td>
<td>-231</td>
<td>1</td>
<td>5.729</td>
<td>1.680</td>
</tr>
<tr>
<td>$x^3 - 4x^2 - 3x - 1$</td>
<td>-247</td>
<td>1</td>
<td>4.686</td>
<td>1.912</td>
</tr>
<tr>
<td>$x^3 - 8x^2 + 5x - 1$</td>
<td>-255</td>
<td>1</td>
<td>7.338</td>
<td>1.547</td>
</tr>
<tr>
<td>$x^3 - 4x^2 + 0x - 1$</td>
<td>-283</td>
<td>1</td>
<td>4.061</td>
<td>2.177</td>
</tr>
<tr>
<td>$x^3 - 4x^2 - 2x - 1$</td>
<td>-331</td>
<td>1</td>
<td>4.495</td>
<td>2.255</td>
</tr>
<tr>
<td>$x^3 - 4x^2 - 1x - 1$</td>
<td>-335</td>
<td>1</td>
<td>4.288</td>
<td>2.315</td>
</tr>
<tr>
<td>$x^3 - 7x^2 + 4x - 1$</td>
<td>-367</td>
<td>1</td>
<td>6.400</td>
<td>2.000</td>
</tr>
<tr>
<td>$x^3 - 5x^2 + 1x - 1$</td>
<td>-416</td>
<td>2</td>
<td>4.836</td>
<td>2.446</td>
</tr>
<tr>
<td>$x^3 - 5x^2 - 3x - 1$</td>
<td>-464</td>
<td>2</td>
<td>5.571</td>
<td>2.414</td>
</tr>
<tr>
<td>$x^3 - 6x^2 - 4x - 1$</td>
<td>-491</td>
<td>1</td>
<td>6.627</td>
<td>2.271</td>
</tr>
<tr>
<td>$x^3 - 5x^2 + 0x - 1$</td>
<td>-527</td>
<td>1</td>
<td>5.040</td>
<td>2.701</td>
</tr>
<tr>
<td>$x^3 - 6x^2 + 2x - 1$</td>
<td>-563</td>
<td>1</td>
<td>5.679</td>
<td>2.634</td>
</tr>
<tr>
<td>$x^3 - 5x^2 - 1x - 1$</td>
<td>-588</td>
<td>2</td>
<td>5.228</td>
<td>2.803</td>
</tr>
<tr>
<td>$x^3 - 9x^2 + 5x - 1$</td>
<td>-608</td>
<td>2</td>
<td>8.421</td>
<td>2.208</td>
</tr>
</tbody>
</table>
### Units of Real Cubic Fields

| \(x^3 - 10x^2 - 6x - 1\) | -643 | 1 | 10.577 | 1.977 |
| \(x^3 - 11x^2 + 6x - 1\) | -671 | 1 | 10.435 | 2.036 |
| \(x^3 - 8x^2 - 5x - 1\) | -695 | 1 | 8.596 | 2.332 |
| \(x^3 - 8x^2 + 4x - 1\) | -731 | 1 | 7.484 | 2.591 |
| \(x^3 - 6x^2 + 1x - 1\) | -751 | 1 | 5.859 | 2.995 |
| \(x^3 - 7x^2 - 4x - 1\) | -863 | 1 | 7.548 | 2.802 |
| \(x^3 - 6x^2 + 0x - 1\) | -891 | 3 | 6.028 | 3.215 |
| \(x^3 - 6x^2 - 2x - 1\) | -931 | 1 | 6.341 | 3.201 |
| \(x^3 - 6x^2 - 1x - 1\) | -959 | 1 | 6.188 | 3.290 |
| \(x^3 - 7x^2 + 2x - 1\) | -983 | 1 | 6.725 | 3.187 |

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### References


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