n = \Delta + \Delta + 2(\Delta + \Delta)

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Abstract

Let \( n \) be a nonnegative integer. An explicit formula is given for the number of quadruples \( (t_1, t_2, t_3, t_4) \) of triangular numbers such that

\[ n = t_1 + t_2 + 2(t_3 + t_4). \]

As a consequence of this formula we deduce that every nonnegative integer is of the form \( t_1 + t_2 + 2(t_3 + t_4) \) for some triangular numbers \( t_1, t_2, t_3, t_4 \).

1. Introduction

Let \( N = \{1, 2, 3, \ldots\} \) and \( N_0 = \{0, 1, 2, 3, \ldots\} \). The triangular numbers are the nonnegative integers

\[ T_k = \frac{1}{2} k(k + 1), \quad k \in N_0, \]

so that

\[ T_0 = 0, \ T_1 = 1, \ T_2 = 3, \ T_3 = 6, \ T_4 = 10, \ T_5 = 15, \ldots. \]

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We set

$$\Delta = \{T_k \mid k \in \mathbb{N}_0\}.$$ 

For $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$ we let

$$\delta_m(n) = \text{card}\{(t_1, \ldots, t_m) \in \Delta^m \mid n = t_1 + \cdots + t_m\},$$

so that $\delta_m(n)$ counts the number of representations of $n$ as the sum of $m$ triangular numbers. It is an easily proved classical result that

$$\delta_2(n) = \sum_{d \mid 4n+1} \left( -\frac{4}{d} \right), \quad n \in \mathbb{N}_0, \quad (1.1)$$

where $d$ runs through the positive integers dividing $4n + 1$ and

$$\left( -\frac{4}{d} \right) = \begin{cases} 
+1, & \text{if } d \equiv 1 \pmod{4}, \\
0, & \text{if } d \equiv 0 \pmod{2}, \\
-1, & \text{if } d \equiv -1 \pmod{4},
\end{cases}$$

see for example [4, pp. 77-78]. Similarly

$$\delta_4(n) = \sigma(2n+1), \quad n \in \mathbb{N}_0, \quad (1.2)$$

where

$$\sigma(m) = \sum_{d \mid m} d, \quad m \in \mathbb{N},$$

$$0, \quad m \notin \mathbb{N}.$$ 

The result (1.2) was known to Legendre [2]. A proof using modular forms has been given by Ono, Robins and Wahl [4, pp. 79-80]. An elementary arithmetic proof has been given by Huard, Ou, Spearman and Williams [1, pp. 259-262].

In this paper, we determine

$$R(n) - \text{card}\{(t_1, t_2, t_3, t_4) \in \Delta^4 \mid n = t_1 + t_2 + 2(t_3 + t_4)\} \quad (1.3)$$

by entirely arithmetic means. We make use of the elementary result (1.1) as well as the recent elementary identity due to Huard, Ou, Spearman
and Williams [1, Theorem 1, p. 230], which was used to prove (1.2). This identity is given in Section 2 as Proposition 1. In Section 3, we prove the following result.

**Theorem.** For \( n \in \mathbb{N}_0 \) we have

\[
R(n) = \frac{1}{4} \sum_{d \mid 4n+3} (d - (-1)^{(d-1)/2}).
\]

An immediate consequence of this theorem is that every nonnegative integer is of the form \( t_1 + t_2 + 2t_3 + 2t_4 \) for triangular numbers \( t_1, t_2, t_3, t_4 \).

### 2. Preliminary Results

The elementary identity of Huard, Ou, Spearman and Williams [1, Theorem 1, p. 230] mentioned in Section 1 is the following result:

**Proposition 1.** Let \( f : \mathbb{Z}^4 \to \mathbb{C} \) be such that

\[
f(a, b, x, y) - f(x, y, a, b) = f(-a, -b, x, y) - f(x, y, -a, -b)
\]

for all \((a, b, x, y) \in \mathbb{Z}^4 \). Then, for \( n \in \mathbb{N} \), we have

\[
\sum_{(a,b,x,y) \in \mathbb{N}^4 \atop ax+by=n} (f(a, b, x, -y) - f(a, -b, x, y) + f(a, a-b, x+y, y)
\]

\[
- f(a, a+b, y-x, y) + f(b-a, b, x, x+y) - f(a+b, b, x, x-y))
\]

\[
= \sum_{d \mid n} \sum_{x=1}^{d-1} \sum_{\substack{(0, n/d, x, d) + f(n/d, 0, d, x) + f(n/d, n/d, d-x, -x)}
- f(x, x-d, n/d, n/d) - f(x, d, 0, n/d) - f(d, x, n/d, 0)).
\]

For \( x \in \mathbb{Z} \) and \( k \in \mathbb{N} \) with \( k \geq 2 \), we let

\[
F_k(x) = \begin{cases} 
1, & \text{if } k \mid x, \\
0, & \text{if } k \nmid x.
\end{cases}
\]
As usual we set

\[ d(m) = \begin{cases} 1, & \text{if } m \in \mathbb{N}, \\ 0, & \text{if } m \notin \mathbb{N}. \end{cases} \]

Taking \( f(a, b, x, y) = f(a)F_2(x) \) in Proposition 1, where \( f : \mathbb{Z} \mapsto \mathbb{C} \) is an even function, we obtain the following interesting identity.

**Proposition 2.** Let \( f : \mathbb{Z} \mapsto \mathbb{C} \) be an even function. Then, for \( n \in \mathbb{N} \), we have

\[
\sum_{(a, b, x, y) \in \mathbb{N}^4 \atop 2ax + by = n} (f(a - b) - f(a + b))
\]

\[
= \frac{1}{2} f(0)(\sigma(n) - d(n) - d(n/2)) + \frac{1}{2} \sum_{d \in \mathbb{N}} \left(1 + \frac{n}{d}\right)f(d)
\]

\[
+ \frac{1}{2} \sum_{d \in \mathbb{N}} \left(1 - 2d + \frac{2n}{d}\right)f(d) - \sum_{d \in \mathbb{N}} \left(\sum_{v=1}^{d} f(v)\right) - \sum_{d \in \mathbb{N}} \left(\sum_{v=1}^{d} f(v)\right).
\]

Proposition 2 is similar to the following identity of Liouville [3, p. 284].

**Proposition 3.** Let \( f : \mathbb{Z} \mapsto \mathbb{C} \) be an even function. Then, for \( n \in \mathbb{N} \),

\[
\sum_{(a, b, x, y) \in \mathbb{N}^4 \atop ax + by = n} (f(a - b) - f(a + b))
\]

\[
= f(0)(\sigma(n) - d(n)) + \sum_{d \in \mathbb{N}} \left(1 - d + \frac{2n}{d}\right)f(d) - 2 \sum_{d \in \mathbb{N}} \left(\sum_{v=1}^{d} f(v)\right).
\]

Taking \( f(x) = F_4(x) \) in Proposition 2, and replacing \( n \) by \( 4n + 3 \), we obtain, as

\[
F_4(a - b) - F_4(a + b) = \left(\frac{-4}{ab}\right), \text{ for all } a, b \in \mathbb{N},
\]

the following result.
\[ n = \Delta + \Delta + 2(\Delta + \Delta) \]

**Proposition 4.** For \( n \in \mathbb{N}_0 \) we have

\[
\sum_{(a, b, x, y) \in \mathbb{N}^4} \left( -\frac{4}{ab} \right) = \frac{1}{4} \sum_{d \in \mathbb{N}} \left( d - \left( -\frac{4}{d} \right) \right).
\]

### 3. Proof of Theorem

Let \( n \in \mathbb{N}_0 \). By (1.1) and (1.3) we have

\[
R(n) = \sum_{m \in \mathbb{N}_0} \delta_2(m) \delta_2(n - 2m). \tag{3.1}
\]

Appealing to (1.1) and (3.1), we obtain

\[
R(n) = \sum_{m \in \mathbb{N}_0} \left( \sum_{a \in \mathbb{N}} \left( -\frac{4}{a} \right) \right) \left( \sum_{b \in \mathbb{N}} \left( -\frac{4}{b} \right) \right). \tag{3.2}
\]

Now

\[
\left( -\frac{4}{a} \right) \left( -\frac{4}{b} \right) = \left( -\frac{4}{ab} \right), \text{ for all } a, b \in \mathbb{N},
\]

and

\[
a | 4m + 1, b | 4(n - 2m) + 1 \text{ for some } m \in \mathbb{N}_0 \text{ with } m \leq n/2
\]

\[
\Leftrightarrow 4n + 3 = 2ax + by, ax \equiv 1 \pmod{4} \text{ for some } x, y \in \mathbb{N}.
\]

Hence (3.2) becomes

\[
R(n) = \sum_{(a, b, x, y) \in \mathbb{N}^4} \left( -\frac{4}{ab} \right). \tag{3.3}
\]

Next, we show that

\[
\sum_{(a, b, x, y) \in \mathbb{N}^4} \left( -\frac{4}{ab} \right) = 0. \tag{3.4}
\]
Interchanging the roles of $a$ and $x$ in the sum (3.4), and noting that
\[
\left(\frac{-4}{xb}\right) = \left(\frac{-4}{ax+3}\right) = \left(\frac{-4}{ax}\right)\left(\frac{-4}{ab}\right) = \left(\frac{-4}{3}\right)\left(\frac{-4}{ab}\right) = -\left(\frac{-4}{ab}\right),
\]
we obtain
\[
\sum_{(a,b,x,y)\in\mathbb{N}^4\atop 2ax+by=4n+3\atop ax=3\pmod{4}} \left(\frac{-4}{ab}\right) = \sum_{(a,b,x,y)\in\mathbb{N}^4\atop 2ax+by=4n+3\atop ax=3\pmod{4}} \left(\frac{-4}{xb}\right) = -\sum_{(a,b,x,y)\in\mathbb{N}^4\atop 2ax+by=4n+3\atop ax=3\pmod{4}} \left(\frac{-4}{ab}\right),
\]
from which (3.4) follows. Adding (3.4) to (3.1), we obtain
\[
R(n) = \sum_{(a,b,x,y)\in\mathbb{N}^4\atop 2ax+by=4n+3\atop x=1\pmod{2}} \left(\frac{-4}{ab}\right). 
\]
(3.5)

If $a$ is even, then $\left(\frac{-4}{ab}\right) = 0$ so
\[
R(n) = \sum_{(a,b,x,y)\in\mathbb{N}^4\atop 2ax+by=4n+3\atop x=1\pmod{2}} \left(\frac{-4}{ab}\right). 
\]
(3.6)

Next, we show that
\[
\sum_{(a,b,x,y)\in\mathbb{N}^4\atop 2ax+by=4n+3\atop x=0\pmod{2}} \left(\frac{-4}{ab}\right) = 0.
\]
(3.7)

Interchanging the roles of $b$ and $y$ in the sum in (3.7), and noting that
\[
\left(\frac{-4}{ay}\right) = \left(\frac{-4}{by}\right)\left(\frac{-4}{ab}\right) = \left(\frac{-4}{3}\right)\left(\frac{-4}{ab}\right) = -\left(\frac{-4}{ab}\right),
\]
we obtain
\[
\sum_{(a,b,x,y)\in\mathbb{N}^4\atop 2ax+by=4n+3\atop x=0\pmod{2}} \left(\frac{-4}{ab}\right) = \sum_{(a,b,x,y)\in\mathbb{N}^4\atop 2ax+by=4n+3\atop x=0\pmod{2}} \left(\frac{-4}{ay}\right) = -\sum_{(a,b,x,y)\in\mathbb{N}^4\atop 2ax+by=4n+3\atop x=0\pmod{2}} \left(\frac{-4}{ab}\right),
\]
\[ n = \Delta + \Delta + 2(\Delta + \Delta) \]

from which (3.7) follows. Adding (3.6) and (3.7), we obtain

\[ R(n) = \sum_{(a, b, x, y) \in \mathbb{N}^4} \left( \frac{-d}{ab} \right). \]  

Appealing to Proposition 4, (3.8) yields

\[ R(n) = \frac{1}{4} \sum_{d \in \mathbb{N}} \left( d - \left( \frac{-4}{d} \right) \right). \]

The theorem now follows as \( \left( \frac{-4}{d} \right) = (-1)^{(d-1)/2} \) for \( d \) odd.

For odd \( d \in \mathbb{N} \) we have

\[ d - (-1)^{(d-1)/2} \geq d - 1 \geq 0. \]

Hence for \( n \in \mathbb{N}_0 \) we deduce that

\[ R(n) = \frac{1}{4} \sum_{d \mid 4n+3} \left( d - (-1)^{(d-1)/2} \right) + n + 1 \geq 0 + 0 + 1 = 1. \]

This shows that every nonnegative integer is of the form \( t_1 + t_2 + 2(t_3 + t_4) \) for some triangular numbers \( t_1, t_2, t_3, t_4 \).

For example, with \( n = 6 \) we have

\[ R(6) = \frac{1}{4} \sum_{d \mid 27} \left( d - (-1)^{(d-1)/2} \right) = \frac{1}{4} (0 + 4 + 8 + 28) = 10. \]

The 10 representations \( (t_1, t_2, t_3, t_4) \in \Delta^4 \) in 6 = \( t_1 + t_2 + 2(t_3 + t_4) \) are

\[
(0,0,0,3), (0,0,3,0), (0,6,0,0), (1,1,1,1), (1,3,0,1), \\
(1,3,1,0), (3,1,0,1), (3,3,0,0), (6,0,0,0).
\]
References


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