DENSITY OF INTEGERS WHICH ARE
DISCRiminANTS OF CYCLIC CUBIC FIELDS

BLAIR K. SPEARMAN and KENNETH S. WILLIAMS

(Received September 11, 2002)

Submitted by K. K. Azad

Abstract

An asymptotic formula is given for the number of integers \( \leq x \) which
are discriminants of cyclic cubic fields.

Let \( n \) be a positive integer. It is known that \( n \) is the discriminant of a
cyclic cubic field if and only if

\[
n = 81, \ (q_1 \cdots q_r)^2 \quad \text{or} \quad 81(q_1 \cdots q_r)^2,
\]

where \( r \) is a positive integer and \( q_1, ..., q_r \) are distinct primes
\( = 1(\text{mod } 3) \), see for example [2], [3]. Let \( A \) denote the set of positive
integers which are the product of distinct primes \( = 1(\text{mod } 3) \) including
the empty product = 1. Then the number \( K(x) \) of \( n \leq x \) which are
discriminants of cyclic cubic fields is for \( x > 81 \times 7^2 \),

---

Mathematics Subject Classification: 11R16, 11R21, 11R29.
Key words and phrases: discriminant, cyclic cubic field.
Both authors were supported by research grants from the Natural Sciences and
Engineering Research Council of Canada.

© 2003 Pushpa Publishing House
\[ K(x) = 1 + \sum_{1 < n \leq x^{1/2}} 1 + \sum_{n \in A} \frac{1}{n^{1/2}/3} \]

so that

\[ K(x) = Q(x^{1/2}) + Q(x^{1/2}/9) - 1, \]

where

\[ Q(x) = \sum_{n \leq x} 1. \]

Our purpose is to determine the behaviour of \( K(x) \) for large \( x \). To do this we make use of a theorem of Wirsing [5], the prime number theorem for the arithmetic progression \( \{3k + 1 : k \in \mathbb{N}\} \), and Mertens' theorem for the arithmetic progression \( \{3k + 1 : k \in \mathbb{N}\} \) [4]. Throughout this paper \( p \) denotes a prime number.

**Wirsing's theorem.** Let \( f(n) \) be a multiplicative function such that

\[ f(n) \geq 0, \quad \text{for } n = 1, 2, 3, \ldots, \]

\[ f(p^k) \leq c_1c_2^k, \quad \text{for constants } c_1 \text{ and } c_2 \text{ with } c_2 < 2 \text{ and } k = 1, 2, 3, \ldots, \]

\[ \sum_{p \leq x} f(p) = (\tau + o(1)) \frac{x}{\log x}, \quad \text{as } x \to \infty, \]

then

\[ \sum_{n \leq x} f(n) = \left( e^{-\gamma} + o(1) \right) \frac{x}{\log x} \prod_{p \leq x} \left( 1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \ldots \right), \]

as \( x \to \infty \), where \( \gamma \) is Euler's constant.

**Prime number theorem for primes** \( p = 1 \pmod{3} \). As \( x \to \infty \),

\[ \sum_{p \leq x \atop p = 1 \pmod{3}} 1 = \left( \frac{1}{2} + o(1) \right) \frac{x}{\log x}. \]
Mertens' theorem for primes $p = 1 \mod 3$. As $x \to \infty$,

$$
\prod_{\substack{p \leq x \\
p = 1 \mod 3}} \left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma/2}2^{1/2} \pi^{1/2}}{3^{1/4}} \prod_{p = 1 \mod 3} \left(1 - \frac{1}{p^2}\right)^{1/2} (\log x)^{-1/2} + O((\log x)^{-3/2}).
$$

(9)

We are now ready to prove

**Theorem 1.** As $x \to \infty$,

$$
Q(x) = \left(\frac{3^{1/4}}{2^{1/2} \pi} \prod_{p = 1 \mod 3} \left(1 - \frac{1}{p^2}\right)^{1/2} + o(1)\right) \frac{x}{\sqrt{\log x}}.
$$

Proof. We let

$$
f(n) = \begin{cases} 
1, & \text{if } n \in A, \\
0, & \text{if } n \notin A.
\end{cases}
$$

Clearly $f(n)$ is a multiplicative function satisfying (4), (5) (with $c_1 = c_2 = 1$) and (6) (with $\tau = \frac{1}{2}$ by (8)). Hence, by Wirsing's theorem, we obtain as $\Gamma(1/2) = \sqrt{\pi}$,

$$
Q(x) = \sum_{n \leq x \atop n \in A} \frac{1}{\sqrt{\pi}} (1 + o(1)) \frac{x}{\log x} \prod_{p \leq x \atop p = 1 \mod 3} \left(1 + \frac{1}{p}\right) \text{ as } x \to \infty.
$$

Next for $x \to \infty$, we have

$$
\prod_{p \leq x \atop p = 1 \mod 3} \left(1 + \frac{1}{p}\right) = \frac{R}{S},
$$

where

$$
R = \prod_{p \leq x \atop p = 1 \mod 3} \left(1 - \frac{1}{p^2}\right) = (1 + o(1)) \prod_{p = 1 \mod 3} \left(1 - \frac{1}{p^2}\right)
$$
and by (9)

\[ S = \prod_{p \equiv 1 (\text{mod } 3)} p \left( 1 - \frac{1}{p} \right)^{1/2} \]

\[ = \frac{e^{-\gamma/2} 2^{1/2} \pi^{1/2}}{3^{1/4}} \prod_{p \equiv 1 (\text{mod } 3)} \left( 1 - \frac{1}{p^2} \right)^{1/2} (1 + o(1)) \frac{1}{\sqrt{\log x}} \]

so that

\[ \prod_{p \leq x \atop p \equiv 1 (\text{mod } 3)} \left( 1 + \frac{1}{p} \right) = \frac{e^{\gamma/2} 3^{1/4}}{2^{1/2} \pi^{1/2}} \prod_{p \equiv 1 (\text{mod } 3)} \left( 1 - \frac{1}{p^2} \right)^{1/2} (1 + o(1)) \frac{\sqrt{x}}{\log x} \]

Finally

\[ Q(x) = \frac{3^{1/4}}{2^{1/2} \pi \frac{x^{1/2}}{\sqrt{\log x}}} \prod_{p \equiv 1 (\text{mod } 3)} \left( 1 - \frac{1}{p^2} \right)^{1/2} (1 + o(1)) \frac{x}{\sqrt{\log x}}, \quad \text{as } x \to \infty. \]

From Theorem 1 and (2), we obtain

**Theorem 2.** As \( x \to \infty \),

\[ K(x) = \frac{3^{1/4}}{\pi} \frac{10}{9} x^{1/2} \prod_{p \equiv 1 (\text{mod } 3)} \left( 1 - \frac{1}{p^2} \right)^{1/2} (1 + o(1)). \]

H. Cohn [1] has shown that the number \( N(x) \) of cyclic cubic fields of discriminant \( \leq x \) satisfies

\[ N(x) = \frac{3^{1/2}}{2 \pi} \frac{11}{18} \prod_{p \equiv 1 (\text{mod } 3)} \frac{(p + 2)(p - 1)}{p(p + 1)} x^{1/2} (1 + o(1)), \]

as \( x \to \infty \).

**References**


Department of Mathematics and Statistics
Okanagan University College
Kelowna, B.C. Canada V1V 1V7
e-mail: bspearnan@okanagan.bc.ca

Centre for Research in Algebra and Number Theory
School of Mathematics and Statistics
Carleton University
Ottawa, Ontario, Canada K1S 5B6
e-mail: williams@math.carleton.ca