CYCLIC QUARTIC FIELDS AND $F_{20}$ QUINTICS

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Abstract

It is shown how to determine the unique quartic subfield of the splitting field of an irreducible quintic polynomial with Galois group $F_{20}$.

Let $f(X) \in \mathbb{Q}[X]$ be a monic solvable irreducible quintic polynomial. As $f(X)$ is solvable its Galois group $G$ is $Z_5$ (the cyclic group of order 5), $D_5$ (the dihedral group of order 10), or $F_{20}$ (the Frobenius group of order 20). Let $L$ denote the splitting field of $f$. If $G = Z_5$, then $L$ does not possess a quadratic subfield. If $G = D_5$, then $L$ possesses a unique quadratic subfield $k$. The determination of this quadratic subfield $k$ has been treated by Jensen and Yui [3, 4], Williamson [7], and by Spearman, Spearman and Williams [6] when $f(X)$ is a trinomial of the form $X^5 + ax + b$. If $G = F_{20}$, then $L$ possesses a unique quadratic subfield $k$ (which must be real) and a unique quartic subfield $K$ (which must be cyclic and contains $k$). It is well known that $k = \mathbb{Q} \left( \sqrt{d} \right)$, where $d(> 0)$ is the discriminant of $f(X)$. When $f(X) = X^5 + ax + b$, Spearman, Spearman and Williams [6] have given an explicit formula for $K$. In this

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paper we show how to determine $K$ for an arbitrary monic irreducible quintic polynomial $f(X)$ with Galois group $G = F_{20}$.

If $n$ is a positive integer, we write $(n)$ to denote a monic irreducible polynomial in $\mathbb{Q}[X]$ of degree $n$, and we set

$$1) \quad S = \{ p(\text{prime}) \mid p \mid d, f(X) = (1)(2)(2)(\text{mod } p) \}.$$ 

Let $p \in S$. The two irreducible quadratics in the factorization of $f(X)(\text{mod } p)$ are distinct (mod $p$) as $p \mid d$. Hence $p \neq 2$ as there is a unique irreducible quadratic polynomial (mod 2) namely $X^2 + X + 1$. Let $D$ be the squarefree part of $d$. By Stickelberger's theorem [5, p. 153], we have

$$2) \quad \left( \frac{D}{p} \right) = \left( \frac{d}{p} \right) = (-1)^{5-3} = 1.$$ 

Hence for $p \in S$ we can let $E_p$ denote an integer such that $D = E_p^2 \pmod{p}$. We prove

**Theorem.** Let $f(X)$ be a monic irreducible quintic polynomial with Galois group $F_{20}$. Let $d > 0$ be the discriminant of $f(X)$. Let $D > 0$ be the squarefree part of $d$. Then there are unique integers $A, B, C$ with the following properties:

$$3) \quad A \text{ is squarefree and odd},$$

$$4) \quad D = B^2 + C^2, \quad B > 0, C > 0,$$

$$5) \quad (A, D) = 1,$$

$$6) \quad A \mid d,$$

$$7) \quad \left( \frac{A(D + BE_p)}{p} \right) = -1 \text{ for all } p \in S \text{ with } p \nmid C.$$ 

Then the unique quartic subfield $K$ of the splitting field $L$ of $f(X)$ is

$$K = \mathbb{Q}\left( \sqrt{A(D + B\sqrt{D})} \right).$$
Proof. The unique quadratic subfield of $L$ is $k = \mathbb{Q}(\sqrt{d}) = \mathbb{Q}(\sqrt{D})$. As $K$ is a cyclic quartic field with quadratic subfield $\mathbb{Q}(\sqrt{D})$, where $D$ is squarefree, there exist unique integers $A, B, C$ satisfying (3), (4), (5) and (8) [2]. Let $\theta$ be a root of $f(X)$ and set $M = \mathbb{Q}(\theta)$ so that $[M : \mathbb{Q}] = 5$. The compositum of $K$ and $M$ is $L$. Hence the set of primes dividing the discriminant $d(L)$ coincides with the set of primes dividing $d(K)d(M)$ [5, p. 167]. But $L$ is the minimal normal extension of $\mathbb{Q}$ containing $M$ so $d(M)$ and $d(L)$ contain the same primes [5, p. 168]. Let $q$ be a prime such that $q \mid A$. As $d(K) = 2^e A^2 D^3$, where $e = 0, 4, 6, 8$, see [2], we have $q \mid d(K)$. Hence $q \mid d(L)$ and so $q \mid d(M)$. But $A$ is squarefree so $A \mid d(M)$. Hence $A \mid d$, which is (6).

Now, let $p \in S$, $p \not\mid C$. An easy calculation shows that $p \mid A(D + BE_p)$. As $\left(\frac{D}{p}\right) = +1$, $p$ splits completely in $k$, say $p = PP'$. The prime ideal $P$ (and similarly for $P'$) splits in $K$ if and only if

$$ \left[ \frac{A(D + B\sqrt{D})}{P} \right]_2 = +1 $$

$$ \Leftrightarrow \left[ \frac{A(D + \varepsilon BE_p)}{P} \right]_2 = +1, \text{ where } \sqrt{D} = \varepsilon E_p \pmod{P}, \varepsilon = \pm 1, $$

$$ \Leftrightarrow \left[ \frac{A(D + BE_p)}{P} \right]_2 = +1, \text{ as } \left[ \frac{A(D + BE_p)}{P} \right]_2 \left[ \frac{A(D - BE_p)}{P} \right]_2 = \left[ \frac{A^2 C^2 E_p^2}{P} \right] = +1, $$

$$ \Leftrightarrow \left( \frac{A(D + BE_p)}{p} \right) = +1. $$

Suppose $\left( \frac{A(D + BE_p)}{p} \right) = +1$. Then, by the above, $P$ and $P'$ split in $K$ so
that \( p \) splits completely in \( K \). As \( L \) is a normal extension of \( K \) of degree 5, \( p \) must factor either as \( P_1 P_2 P_3 P_4 \) with each \( N(P_i) = p^5 \) or as \( P_1 P_2 \cdots P_{20} \) with each \( N(P_i) = p \). Now as \( p \in S \), we have \( p = Q_1 Q_2 Q_3 \) in \( M \) with \( N(Q_1) = p \), \( N(Q_2) = N(Q_3) = p^2 \). Since \( L \) is a quadratic extension of a quadratic extension of \( M \), the prime ideal factors of \( Q_2 \) in \( L \) have norms \( p^2 \), \( p^4 \) or \( p^8 \), a contradiction. Hence

\[
\frac{A(D + BE_p)}{p} = -1.
\]

This theorem can easily be put in the form of an algorithm to determine the unique quartic subfield of the splitting field of a given irreducible quintic polynomial with Galois group \( F_{20} \).

INPUT. \( f(X) \)--irreducible quintic with Galois group \( F_{20} \).

STEP 1. Calculate discriminant \( d \) of \( f(X) \).

STEP 2. Calculate squarefree part \( D \) of \( d \).

STEP 3. Determine all pairs of positive integers \((B, C)\) such that

\[
\]

STEP 4. Determine all odd squarefree divisors \( A \) of \( d \) which are coprime with \( D \).

STEP 5. For \( p = 3, 5, 7, 11, \ldots \) with \( p \mid dC \)

factor \( f(X) \pmod{p} \)

if \( f(X) \equiv (1) (2) (2) \pmod{p} \)

eliminate \((A, B, C)\) for which

\[
\frac{A(D + BE_p)}{p} = 1
\]

next \( p \)

else

next \( p \)
OUTPUT. Stop when a single triple \((A, B, C)\) remains.

Required quartic field is \(\mathbb{Q}\left(\sqrt[4]{A(D + B\sqrt{D})}\right)\).

Example.

\[ f(X) = X^5 + 250X^2 + 625 \]

\[ d = 5^{19} \cdot 59^2 \]

\[ D = 5 \]

\( (B, C) = (1, 2), (2, 1) \)

\[ A = \pm 1, \pm 59 \]

primes \(p\) for which \(X^5 + 250X^2 + 625 \equiv (1)(2)(3) \pmod{p}\)

are \(p = 19, 29, 79, 89, \ldots\)

\(p = 19\) eliminates \((A, B, C) = (-1, 1, 2), (1, 2, 1), (-59, 2, 1), (59, 1, 2)\)

\(p = 29\) eliminates \((A, B, C) = (1, 1, 2), (-59, 1, 2)\)

\(p = 89\) eliminates \((A, B, C) = (59, 2, 1)\)

surviving \((A, B, C)\) is \((-1, 2, 1)\).

Hence the unique quartic subfield of the splitting field of \(X^5 + 250X^2 + 625\) is \(\mathbb{Q}\left(\sqrt[4]{5 + 2\sqrt{5}}\right)\).

References


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