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Analysis

SOME RESULTS ON THE GENERALIZED STIELTJES CONSTANTS

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Abstract: In a neighbourhood of $s = 1$, the Hurwitz zeta function $\zeta(s,a)$ has the Laurent expansion in powers of $s-1$

$$\zeta(s,a) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n(a)}{n!}(s-1)^n,$$

where the quantity $\gamma_n(a)$ is the generalized Stieltjes constant. A number of results about the generalized Stieltjes constants are proved.

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1. Introduction. For $0 < a \leq 1$ and $\sigma > 1$ the Hurwitz zeta function $\zeta(s,a)$ is defined by

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \quad s = \sigma + it.$$

It is well-known (see [2, p.255] that $\zeta(s,a)$ has an analytic continuation to the whole complex plane except for a simple pole at $s = 1$ with residue 1. In a neighbourhood of $s = 1$, $\zeta(s,a)$ has the Laurent expansion in powers of $s-1$

$$\zeta(s,a) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n(a)}{n!}(s-1)^n, \tag{1.1}$$

where the quantities $\gamma_n(a)$ are known as the generalized Stieltjes constants. When $a = 1$ $\zeta(s,a)$ reduces to the Riemann zeta function $\zeta(s)$ and (1.1) becomes

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n}{n!}(s-1)^n, \tag{1.2}$$

where $\gamma_n = \gamma_n(1)$ is the (ordinary) Stieltjes constant. Briggs and Chowla [3] and Hardy [8] have given expressions for γ_n. Berndt [4] has used the same technique as in [3] to derive the following expression for $\gamma_n(a)$: for $n = 0, 1, 2, \ldots$

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$$\gamma_n(a) = \lim_{m\to\infty}\Big(\sum_{k=0}^{m}\frac{\log^n(k+a)}{k+a} - \frac{\log^{n+1}(m+a)}{n+1}\Big). \tag{1.3}$$

Wilton [14] has given a similar formula. In Section 4 we prove the following new formula for $\gamma_n(a)$.

THEOREM 1. For $0 < a \le 1$ and $m, n = 0, 1, 2, \ldots$ we have

$$\gamma_n(a) = \sum_{k=0}^{m}\frac{\log^n(k+a)}{k+a} - \frac{\log^{n+1}(m+a)}{n+1} - \frac{\log^n(m+a)}{2(m+a)} + \int_m^\infty P_1(x)f_n(x)dx, \tag{1.4}$$

where $f_n(x) = \frac{\log^n(x+a)}{x+a}$ and $P_1(x) = x - [x] - \frac{1}{2}$.

We remark that the formula (1.3) follows from (1.4) by letting $m \to +\infty$. The formula (1.4) has the advantage that it can be used to estimate the size of $\gamma_n(a)$. In addition Theorem 1 can be used to obtain an asymptotic formula for the sum $\sum_{k=0}^{m}\frac{\log^n(k+a)}{k+a}$ valid for $m \to +\infty$. In Section 5 the following theorem is proved.

THEOREM 2. For $0 < a \le 1$ and $n = 0, 1, 2, \ldots$ we have

$$\sum_{k=0}^{m}\frac{\log^n(k+a)}{k+a} = \frac{\log^{n+1}(m+a)}{n+1} + \gamma_n(a) + \frac{\log^n(m+a)}{2(m+a)} + O\Big(\frac{\log^n m}{m^2}\Big), \tag{1.5}$$

as $m \to +\infty$, where the constant implied by the O-symbol depends only on n.

Berndt [4] has proved the following inequality for the generalized Stieltjes constants: for $n \ge 1$ and $0 < a \le 1$

$$\Big|\gamma_n(a) - \frac{\log^n a}{a}\Big| \le \frac{(3+(-1)^n)(n-1)!}{\pi^n}. \tag{1.6}$$

It should be noted that Berndt's definition of the Stieltjes constant is a little different from ours. In Section 6 we improve the inequality (1.6). We prove

THEOREM 3. Let $0 < a \le 1$. For $n \ge 1$ we have

$$\Big|\gamma_n(a) - \frac{\log^n a}{a}\Big| \le \frac{(3+(-1)^n)(2n)!}{n^{n+1}(2\pi)^n}, \tag{1.7}$$

and for $n = 0$

$$\Big|\gamma_0(a) - \frac{1}{a}\Big| \le \gamma = 0.577\ldots\ ,$$

where γ denotes Euler's constant.

Liang and Todd [12] have expressed the Stieltjes constants $\gamma_n = \gamma_n(1)$ in terms of the Bernoulli numbers $B_k (k = 0, 1, 2, \ldots)$, where $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}, \ldots$, and the quantities $\tau_k (k = 0, 1, 2, \ldots)$ defined by

$$\tau_k = \sum_{n=1}^{\infty} \frac{(-1)^n \log^k n}{n}, \quad k \geq 1,$$
$$\tau_0 = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -\log 2.$$

They proved, for $k = 0, 1, 2, \ldots$, that [12, pp.168-169]

$$\gamma_k = \frac{(\log 2)^{k+1}}{k+1} \sum_{j=1}^{k+1} \frac{\binom{k+1}{j} B_{k+1-j}}{j+1} + \frac{1}{k+1} \sum_{j=1}^{k+1} \binom{k+1}{j} B_{k+1-j} (\log 2)^{k-j} \tau_j. \tag{1.8}$$

We show in Section 7 that this formula can be simplified by recognizing the first sum as being the term $j = 0$ of the second sum.

THEOREM 4. For $k = 0, 1, 2, \ldots$ we have

$$\gamma_k = \frac{1}{k+1} \sum_{j=0}^{k+1} \binom{k+1}{j} B_{k+1-j} (\log 2)^{k-j} \tau_j. \tag{1.9}$$

Keiper [10] has given a recurrence relation for $\frac{\gamma_n}{n!}$ which involves the values of the Hurwitz zeta function at $s = 3/2$. We show that this recurrence relation is unnecessarily complicated by proving in Section 8 the following simpler relation, which involves the Riemann zeta function $\zeta(n)$ instead of $\zeta\left(n, \frac{3}{2}\right)$.

THEOREM 5. For $n = 0, 1, 2, 3, \ldots$ we have

$$\frac{\gamma_n}{n!} = \frac{1}{n+1} \left\{ a_n - \sum_{j=1}^{n} \frac{\gamma_{j-1} a_{n-j}}{(j-1)!} \right\}, \tag{1.10}$$

with

$$\begin{cases} a_n = \zeta(n+1)\left(1 - \frac{1}{2^n}\right) + \sigma_{n+1} - 1, & n \geq 1, \\ a_0 = \gamma, \end{cases}$$

and

$$\sigma_n = \sum_{\rho} \frac{1}{\rho^n}, \quad n \geq 1,$$

where ρ runs through the nontrivial zeros of the Riemann zeta function (suitably paired when $n = 1$ to ensure convergence).

This theorem leads to the Laurent expansion of $\frac{\zeta'(s)}{\zeta(s)}$ in the form

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + \gamma + \sum_{n=1}^{\infty}(-1)^n\Big\{\zeta(n+1)\Big(1-\frac{1}{2^n}\Big) + \sigma_{n+1} - 1\Big\}(s-1)^n. \tag{1.11}$$

2. An integral representation of $\zeta(s,a)$. We need the following form of the Euler-Maclaurin formula:

$$\sum_{k=m}^{n} f(k) = \int_m^n f(x)dx + \sum_{k=1}^{q}(-1)^k\frac{B_k}{k!}f^{(k+1)}(x)\Big|_m^n + (-1)^{q+1}\int_m^n P_q(x)f^{(q)}(x)dx, \tag{2.1}$$

where q is a fixed positive integer, $f(x) \in C^q[m,n]$, and $P_k(x)$ is the k-th periodic Bernoulli function defined by

$$P_k(x) = \frac{1}{k!}B_k(x-[x]),$$

where $B_k(x)$ is the k-th Bernoulli polynomial (see [9, p.490]).

Taking $f(x) = \frac{1}{(x+a)^s}, 0 < a \le 1$, $\sigma = Re(s) > 1$, $q = 1$ and letting $n \to \infty$ in (2.1), we have

$$\zeta(s,a) = \sum_{k=0}^{m}\frac{1}{(k+a)^s} + \frac{(m+a)^{1-s}}{s-1} - \frac{(m+a)^{-s}}{2} - s\int_m^\infty \frac{P_1(x)}{(x+a)^{s+1}}dx,$$

where

$$P_1(x) = B_1(x-[x]) = x - [x] - \frac{1}{2}.$$

Since $P'_{k+1}(x) = P_k(x)$ $(k \ge 1)$ (see [1, 23.1.5]) we have

$$\int_m^\infty \frac{P_1(x)}{(x+a)^{s+1}}dx = \frac{P_1(x)}{(x+a)^{s+1}}\Big|_m^\infty + (s+1)\int_m^\infty \frac{P_2(x)}{(x+a)^{s+2}}dx.$$

In view of the boundedness of $P_2(x)$, the function defined by the last integral is analytic in the half plane $Re(s) > -1$, that means

$$\zeta(s,a) = \sum_{k=0}^{m}\frac{1}{(k+a)^s} + \frac{(m+a)^{1-s}}{s-1} - \frac{(m+a)^{-s}}{2} - s\int_m^\infty \frac{P_1(x)}{(x+a)^{s+1}}dx, \ \sigma > -1. \tag{2.2}$$

In particular, taking $m = 0$ in (2.2) gives

$$\zeta(s,a) = \frac{a^{1-s}}{s-1} + \frac{a^{-s}}{2} - s\int_0^\infty \frac{P_1(x)}{(x+a)^{s+1}}dx, \ \sigma > -1, \tag{2.3}$$

which appears for example in Berndt [4, (2.3)]. We make use of (2.2) in Section 4.

3. The Hurwitz formula. In [4, Theorem 4] Berndt gives a simple proof of the Hurwitz formula, namely, for $\sigma < 0$

$$\zeta(s,a) = 2^s\pi^{s-1}\Gamma(1-s)\Big\{ \sin\frac{\pi s}{2}\sum_{n=1}^{\infty}\frac{\cos 2n\pi a}{n^{1-s}} + \cos\frac{\pi s}{2}\sum_{n=1}^{\infty}\frac{\sin 2n\pi a}{n^{1-s}}\Big\}. \tag{3.1}$$

Here we note that Berndt's proof can be made even simpler. Since

$$s\int_{-a}^{0}\frac{P_1(x)}{(x+a)^{s+1}}dx = -\Big(\frac{a^{1-s}}{s-1}+\frac{a^{-s}}{2}\Big), \text{ when } -1<\sigma<0,$$

we have

$$\zeta(s,a) = -s\int_{-a}^{\infty}\frac{P_1(x)}{(x+a)^{s+1}}dx = -s\int_{0}^{\infty}\frac{P_1(x-a)}{x^{s+1}}dx, \; -1<\sigma<0. \tag{3.2}$$

From (3.2) and the Fourier expansion of $P_1(x)$

$$P_1(x) = x - [x] - \frac{1}{2} = -\sum_{n=1}^{\infty}\frac{\sin 2n\pi x}{n\pi}, \quad x \neq \text{ integer}.$$

we obtain for $-1<\sigma<0$

$$\begin{aligned}
\zeta(s,a) &= s\sum_{n=1}^{\infty}\frac{1}{n\pi}\int_0^{\infty}\frac{1}{x^{s+1}}\Big\{\cos 2n\pi a\sin 2n\pi x - \sin 2n\pi a\cos 2n\pi x\Big\}dx \\
&= s\sum_{n=1}^{\infty}\Big\{\frac{\cos 2n\pi a}{n\pi}\int_0^{\infty}\frac{\sin 2n\pi a}{x^{s+1}}dx - \frac{\sin 2n\pi a}{n\pi}\int_0^{\infty}\frac{\cos 2n\pi a}{x^{s+1}}dx\Big\} \\
&= -s2^s\pi^{s-1}\Gamma(-s)\sum_{n=1}^{\infty}\frac{1}{n^{1-s}}\Big\{\sin\frac{\pi s}{2}\cos 2n\pi a + \cos\frac{\pi s}{2}\sin 2n\pi a\Big\}.
\end{aligned}$$

which is (3.1). The inversion of summation and integration can be justified as in [13, p.15]. Since the two series in (3.1) converge for $\sigma < 0$, the formula (3.1) holds for all $\sigma < 0$. A proof of (3.1) using the evaluation of a loop integral can be found in [13, p.37] or [2, pp.257-259].

4. The generalized Stieltjes constants $\gamma_n(a)$. In this section, we consider the Laurent expansion of the Hurwitz zeta function at $s = 1$. It is well-known that $\zeta(s,a)$ is a meromorphic function and its only pole is a simple pole at $s=1$ with residue 1. In the neighbourhood of $s = 1$,

$$\zeta(s,a) = \frac{1}{s-1} + \sum_{n=0}^{\infty}\frac{(-1)^n\gamma_n(a)}{n!}(s-1)^n, \tag{4.1}$$

where the $\gamma_n(a)$ are called generalized Stieltjes constants. If $a = 1$, $\zeta(s,a)$ reduces to $\zeta(s)$ and (4.1) becomes

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n}{n!}(s-1)^n. \tag{4.2}$$

We follow the method given in [15] to obtain a new formula for $\gamma_n(a)$, see Theorem 1.

PROOF OF THEOREM 1. We expand each term on the right side of (2.2) in powers of $s-1$:

$$\frac{1}{(k+a)^s} = \frac{1}{k+a} \sum_{n=0}^{\infty} \frac{(-1)^n \log^n(k+a)}{n!}(s-1)^n,$$
$$\frac{(m+a)^{1-s}}{s-1} = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \log^{n+1}(m+a)}{(n+1)!}(s-1)^n,$$
$$\frac{(m+a)^{-s}}{2} = \frac{1}{2(m+a)} \sum_{n=0}^{\infty} \frac{(-1)^n \log^n(m+a)}{n!}(s-1)^n,$$

then

$$\sum_{k=0}^{m} \frac{1}{(k+a)^s} + \frac{(m+a)^{1-s}}{s-1} - \frac{(m+a)^{-s}}{2} = \frac{1}{s-1}$$
$$+ \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{m} \frac{\log^n(k+a)}{k+a} - \frac{\log^{n+1}(m+a)}{n+1} - \frac{\log^n(m+a)}{2(m+a)} \right\} \frac{(-1)^n (s-1)^n}{n!}. \tag{4.3}$$

Next, set

$$g(s) = s \int_m^{\infty} \frac{P_1(x)}{(x+a)^{s+1}} dx, \quad \sigma > -1. \tag{4.4}$$

We have

$$g^{(n)}(s) = s(-1)^n \int_m^{\infty} \frac{P_1(x) \log^n(x+a)}{(x+a)^{s+1}} dx + n(-1)^{n-1} \int_m^{\infty} \frac{P_1(x) \log^{n-1}(x+a)}{(x+a)^{s+1}} dx,$$
$$g^{(n)}(1) = (-1)^{n-1} \int_m^{\infty} P_1(x) f_n'(x) dx.$$

Theorem 1 follows immediately from (4.1), (2.2) and (4.3).

REMARK 1. Letting $m \to +\infty$ in (4.3), we obtain

$$\gamma_n(a) = \lim_{m\to\infty} \left\{ \sum_{k=0}^{m} \frac{\log^n(k+a)}{k+a} - \frac{\log^{n+1}(m+a)}{n+1} \right\}, \quad n = 0, 1, 2, \ldots, \tag{4.5}$$

which appeared in [4, Theorem 3], and with $a = 1$

$$\gamma_n = \gamma_n(1) = \lim_{m\to\infty} \left\{ \sum_{k=1}^{m} \frac{\log^n k}{k} - \frac{\log^{n+1} m}{n+1} \right\}, \quad n = 0, 1, 2, \ldots . \tag{4.6}$$

In particular,

$$\gamma_0(a) = \lim_{m\to\infty}\Big\{\sum_{k=0}^{m}\frac{1}{k+a} - \log(m+a)\Big\} = -\psi(a) \tag{4.7}$$

(see [7, 8.362,2]) and

$$\gamma_0 = \gamma = \lim_{m\to\infty}\Big\{\sum_{k=1}^{m}\frac{1}{k} - \log m\Big\} \tag{4.8}$$

is Euler's constant.

5. A generalization of Euler's asymptotic formula. We note that

$$P_2(x) = O(1), f_n'(x) = O\Big(\frac{\log^n x}{x^2}\Big),\ f_n''(x) = O\Big(\frac{\log^n x}{x^3}\Big), \text{ as } x \to +\infty,$$

where the constants implied by the O-symbols depend at most on n. Then, integrating by parts, we obtain as $P_2'(x) = P_1(x)$

$$\int_m^\infty P_1(x)f_n'(x)dx = O\Big(\frac{\log^n m}{m^2}\Big).$$

This completes the proof of Theorem 2.

6. Estimation of the quantities $C_n(a) = \gamma_n(a) - \frac{\log^n a}{a}$. Taking $m = 0$ in (4.3) gives

$$\gamma_n(a) = \frac{\log^n a}{2a} - \frac{\log^{n+1} a}{n+1} + \int_0^\infty P_1(x)f_n'(x)dx, \tag{6.1}$$

where

$$f_n(x) = \frac{\log^n(x+a)}{x+a}, \quad n = 0,1,2,\dots .$$

Since for $n \geq 1$

$$\int_0^{1-a} P_1(x)f_n'(x)dx = \int_0^{1-a}\Big(x - \frac{1}{2}\Big)f_n'(x)dx = \frac{\log^n a}{2a} + \frac{\log^{n+1} a}{n+1},$$

we have

$$\gamma_n(a) = \frac{\log^n a}{a} + \int_{1-a}^\infty P_1(x)f_n'(x)dx,$$

that is

$$\gamma_n(a) = \frac{\log^n}{a} + \int_1^\infty P_1(x-a)h_n'(x)dx, \tag{6.2}$$

where

$$h_n(x) = \frac{\log^n x}{x}.$$

For $n = 0$ in (6.1), we have

$$\gamma_0(a) = \frac{1}{2a} - \log a - \int_0^\infty \frac{P_1(x)}{(x+a)^2}dx. \tag{6.3}$$

Comparing this with (4.7) gives

$$\psi(a) = -\frac{1}{2a} + \log a + \int_0^\infty \frac{P_1(x)}{(x+a)^2}dx. \tag{6.4}$$

Noticing that $h_n^{(k)}(\infty) = h_n^{(k)}(1) = 0$, $0 \le k \le n-1$, $n \ge 1$ and integrating by parts $n-1$ times, we obtain for $n \ge 1$

$$\gamma_n(a) = \frac{\log^n a}{a} + (-1)^{n-1} \int_1^\infty P_1(x-a) h_n^{(n)}(x)dx. \tag{6.5}$$

Hence, it is natural for $0 < a \le 1$ to introduce the quantities

$$C_n(a) = \gamma_n(a) - \frac{\log^n a}{a}, \quad n = 1, 2, 3, \dots, \tag{6.6}$$

and

$$C_0(a) = \gamma_0(a) - \frac{1}{a}. \tag{6.7}$$

In [4, Theorem 2], Berndt proved for $0 < a \le 1$ and $n \ge 1$ that

$$|C_n(a)| \le (3 + (-1)^n)\frac{(n-1)!}{\pi^n}. \tag{6.8}$$

However, we can improve this inequality.

PROOF OF THEOREM 3. From (4.7) and (6.7)

$$C_0(a) = \gamma_0(a) - \frac{1}{a} = -\psi(a) - \frac{1}{a} = -\psi(1+a), \text{ where } \psi(a) = \frac{\Gamma'(a)}{\Gamma(a)}, \tag{6.9}$$

see [1, 6.3.5]. Since $\psi(x)$ is increasing on the interval [1,2] and $\psi(1) = -\gamma$, $\psi(2) = 1-\gamma$, we have

$$|C_0(a)| \le \gamma = 0.577\dots, \tag{6.10}$$

which is the second inequality in Theorem 3. Next we prove (1.7). Since

$$h_n(x) = \log^n x/x = \frac{1}{n+1}(\log^{n+1} x)',$$

we have from (6.5)

$$C_n(a) = \frac{(-1)^{n-1}}{n+1}\int_1^\infty P_1(x-a)(\log^{n+1} x)^{(n+1)}dx, \quad n \geq 1. \tag{6.11}$$

According to the result in [11, Lemma]

$$(\log^n x)^{(n)} = \frac{n!}{x^n}\sum_{k=0}^{n}\frac{\log^k x}{k!}s(n, n-k), \tag{6.12}$$

where $s(n,k)$ is the Stirling number of the first kind, we obtain

$$C_n(a) = (-1)^{n-1}n!\sum_{k=0}^{n+1}\frac{s(n+1, n+1-k)}{k!}\int_1^\infty P_n(x-a)\frac{\log^k x}{x^{n+1}}dx, \quad n \geq 1.$$

In view of the fact (see [4, (3.7)])

$$|P_n(x)| \leq \frac{(3+(-1)^n)}{(2\pi)^n}, \quad n \geq 1, \tag{6.13}$$

we obtain

$$\begin{aligned}|C_n(a)| &\leq \frac{(3+(-1)^n)}{(2\pi)^n}n!\sum_{k=0}^{n+1}\frac{|s(n+1,n+1-k)|}{k!}\int_1^\infty \frac{\log^k x}{x^{n+1}}dx \\ &= \frac{(3+(-1)^n)}{(2\pi)^n}n!\sum_{k=0}^{n+1}\frac{|s(n+1,n+1-k)|}{n^{k+1}} \\ &= \frac{(3+(-1)^n)n!}{(2\pi)^n\cdot n^{n+2}}\sum_{k=0}^{n+1}|s(n+1,k)|n^k.\end{aligned}$$

Now, using the formula (see [5, p.213])

$$\sum_{k=0}^{n}|s(n,k)|x^k = x(x+1)(x+2)\dots(x+n-1) \tag{6.14}$$

we have

$$|C_n(a)| \leq \frac{(3+(-1)^n)(2n)!}{(2\pi)^n n^{n+1}}, \quad n \geq 1.$$

REMARK 2. Since for $n = 1, 2, 3, \dots$

$$\frac{(2n)!}{2^n n^n} \leq n!,$$

our estimation (6.10) is better than Berndt's. Furthermore, from Stirling's asymptotic formula for $n!$, we have

$$\frac{(2n)!}{n^{n+1}(2\pi)^n}\Big/\frac{n!}{n\pi^n} = \frac{(2n)!}{n!2^n n^n} \sim \sqrt{2}\Big(\frac{2}{e}\Big)^n, \text{ as } n \to \infty.$$

REMARK 3. Since

$$\zeta(s,a) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n(a)}{n!}(s-1)^n,$$

$$a^{-s} = \frac{1}{a}e^{-(s-1)\log a} = \sum_{n=0}^{\infty} \frac{(-1)^n \log^n a}{n! a}(s-1)^n,$$

we have

$$\zeta(s,a) - a^{-s} = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n C_n(a)}{n!}(s-1)^n. \tag{6.15}$$

In order to prove

$$\sum_{n=0}^{\infty} \left| \frac{C_n(a)}{n!} \right| < 1 \tag{6.16}$$

and then to prove that $\zeta(s,a) - a^{-s}$ has no zeros on the closed disk $|s-1| \leq 1$, Berndt [4] proved the estimation $|C_0(a)| \leq 0.617$ (as well as estimates for $C_1(a)$ and $C_2(a)$). We observe that $C_0(a) = -\psi(1+a)$ and $|C_0(a)| \leq \gamma$.

7. A linear relationship between γ_n and τ_n. The following result is due to Briggs and Chowla [3]. Another proof has been given in [15].

PROPOSITION 1. For $k = 0, 1, 2, \ldots$ we have

$$\tau_k = -\frac{(\log 2)^{k+1}}{k+1} + \sum_{j=0}^{k-1} \binom{k}{j} (\log 2)^{k-j} \gamma_j. \tag{7.1}$$

Theorem 4 follows by inverting the relation (7.1).

PROOF OF THEOREM 4. We set

$$L(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}, \quad Re(s) > 0. \tag{7.2}$$

Since

$$L(s) = (2^{1-s} - 1)\zeta(s), \tag{7.3}$$

$L(s)$ can be continued analytically to the whole complex plane. Consider the power series expansion of $L(s)$ at $s = 1$,

$$L(s) = \sum_{k=0}^{\infty} \frac{L^{(k)}(1)}{k!}(s-1)^k. \tag{7.4}$$

Differentiating (7.2) k times we obtain

$$L^{(k)}(s) = (-1)^k \sum_{n=1}^{\infty} \frac{(-1)^n \log^k n}{n^s}, \quad Re(s) > 0,$$

and

$$L^{(k)}(1) = (-1)^k \sum_{n=1}^{\infty} \frac{(-1)^n \log^k n}{n}.$$

For $k = 1, 2, 3, \ldots$ we set

$$\tau_k = \sum_{n=1}^{\infty} \frac{(-1)^n \log^k n}{n} \text{ and } \tau_0 = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -\log 2, \tag{7.5}$$

so that

$$L(s) = \sum_{k=0}^{\infty} \frac{(-1)^k \tau_k}{k!}(s-1)^k.$$

On the other hand, we have

$$\frac{1}{2^{1-s}-1} = \frac{1}{e^{-(s-1)\log 2}-1} = \frac{-1}{(s-1)\log 2} + \sum_{k=0}^{\infty} \frac{(-1)^k B_{k+1} \log^k 2}{(k+1)!}(s-1)^k, \ |s-1| < \frac{2\pi}{\log 2}.$$

From (7.3) we obtain

$$\begin{aligned} &\frac{1}{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k \gamma_k}{k!}(s-1)^k \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k \tau_k}{k!}(s-1)^k \Big[-\frac{1}{(s-1)\log 2} + \sum_{k=0}^{\infty} \frac{(-1)^k B_{k+1} \log^k 2}{(k+1)!}(s-1)^k \Big], \end{aligned}$$

and comparing the coefficients on both sides gives Theorem 4.

REMARK 4. Liang and Todd [12] set

$$A = (a_{ij})_{n \times n}, \text{ where } a_{ij} = \begin{cases} \binom{i}{j-1}, & 1 \le j \le i, \\ 0, & j > i. \end{cases}$$

and inverted the matrix form of (7.2), namely

$$A \begin{pmatrix} \gamma_0 \\ \gamma_1/\log 2 \\ \gamma_2/\log^2 2 \\ \vdots \\ \gamma_{n-1}/\log^{n-1} 2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/3 \\ \vdots \\ \vdots \\ 1/(n+1) \end{pmatrix} \log 2 + \begin{pmatrix} \tau_1/\log 2 \\ \tau_2/\log^2 2 \\ \vdots \\ \tau_n/\log^n 2 \end{pmatrix}$$

to obtain

$$\begin{pmatrix} \gamma_0 \\ \gamma_1/\log 2 \\ \vdots \\ \gamma_{n-1}/\log^{n-1} 2 \end{pmatrix} = A^{-1} \begin{pmatrix} 1/2 \\ 1/3 \\ \vdots \\ 1/(n+1) \end{pmatrix} \log 2 + A^{-1} \begin{pmatrix} \tau_1/\log 2 \\ \tau_2/\log^2 2 \\ \vdots \\ \tau_n/\log^n 2 \end{pmatrix}.$$

where

$$A^{-1} = (b_{ij})_{n\times n}, \; b_{ij} = \begin{cases} 0, & i < j, \\ \frac{1}{i}\binom{i}{j} B_{i-j}, & i \geq j, \end{cases}$$

so that

$$\gamma_k = \frac{(\log 2)^{k+1}}{k+1} \sum_{j=1}^{k+1} \binom{k+1}{j} \frac{B_{k+1-j}}{j+1} + \frac{1}{k+1} \sum_{j=1}^{k+1} \binom{k+1}{j} B_{k+1-j} (\log 2)^{k-j} \tau_j. \tag{7.6}$$

We show that the forms (7.2) and (7.6) are in fact the same. This follows from the next proposition which we have been unable to locate in the literature on Bernoulli numbers.

PROPOSITION 2. For $k = 1, 2, \ldots$, we have

$$\sum_{j=0}^{k} \frac{\binom{k}{j}}{j+1} B_{k-j} = 0. \tag{7.7}$$

PROOF. Recall ([1, p.804, 23.1.7])

$$B_k(x+h) = \sum_{j=0}^{k} \binom{k}{j} B_j(x) h^{k-j}.$$

Integrating this equality gives

$$\int_0^h B_k(x+h)dh = \sum_{j=0}^{k} \binom{k}{j} B_j(x) \frac{1}{k-j+1} h^{k-j+1}.$$

In view of

$$B'_{k+1}(x) = (k+1)B_k(x)$$

we have

$$\int_0^h B_k(x+h)dh = \frac{1}{k+1}[B_{k+1}(x+h) - B_{k+1}(x)].$$

Hence

$$\frac{1}{k+1}[B_{k+1}(x+h) - B_{k+1}(x)] = \sum_{j=0}^{k} \binom{k}{j} \frac{B_j(x)}{k-j+1} h^{k-j+1}.$$

Taking $x = 0, \; h = 1$ and observing that $B_k(0) = B_k$, $B_k(1) = (-1)^k B_k(0)$, we obtain

$$\sum_{j=0}^{k} \binom{k}{j} \frac{B_{k-j}}{j+1} = \frac{1}{k+1}[B_{k+1}(1) - B_{k+1}(0)] = 0.$$

8. The relation between γ_k and σ_k. It is well-known that the Riemann function $\xi(s)$ is an entire function and has the infinite product expansion

$$\xi(s) = \frac{1}{2}e^{b_0 s}\prod_{\rho}\Big(1-\frac{s}{\rho}\Big)e^{s/\rho}, \tag{8.1}$$

where

$$b_0 = \frac{1}{2}\log\pi + \log 2 - \frac{\gamma}{2} - 1, \tag{8.2}$$

and ρ runs through the nontrivial zeros of $\zeta(s)$ (see [13, pp.30-31]). From (8.1) and the functional equation for $\xi(s)$:

$$\xi(s) = \xi(1-s), \tag{8.3}$$

we have

$$\xi(1) = \xi(0) = \frac{1}{2}. \tag{8.4}$$

Taking the logarithmic derivative of (8.1) gives

$$\frac{\xi'(s)}{\xi(s)} = b_0 + \sum_{\rho}\Big(\frac{1}{\rho} + \frac{1}{s-\rho}\Big). \tag{8.5}$$

Taking $s = 0$ in (8.5) we obtain from (8.3) and (8.4)

$$\xi'(1) = \xi'(0) = -\frac{b_0}{2}. \tag{8.6}$$

Since $1-\rho$ is also a zero of $\zeta(s)$, we obtain from (8.5) with $s = 1$

$$2\sum_{\rho}\frac{1}{\rho} = \frac{\xi'(1)}{\xi(1)} - b_0,$$

that is

$$\sum_{\rho}\frac{1}{\rho} = -b_0, \tag{8.7}$$

which was given by Davenport in [6, pp.81-82].

From (8.5) and (8.7), we have

$$\frac{\xi'(s)}{\xi(s)} = b_0 + \sum_{\rho}\Big(\frac{1}{\rho} - \frac{1}{\rho}\cdot\frac{1}{1-\frac{s}{\rho}}\Big) = b_0 + \sum_{\rho}\Big(\frac{1}{\rho} - \sum_{k=0}^{\infty}\frac{s^k}{\rho^{k+1}}\Big),$$

that is

$$\frac{\xi'(s)}{\xi(s)} = -\sum_{k=0}^{\infty}\sum_{\rho}\frac{1}{\rho^{k+1}}s^k.$$

If we set

$$\sigma_k = \sum_\rho \frac{1}{\rho^k}, \quad k = 1, 2, 3, \ldots, \tag{8.8}$$

then, we have

$$\frac{\xi'(s)}{\xi(s)} = -\sum_{k=0}^{\infty} \sigma_{k+1} s^k \tag{8.9}$$

and by (8.3)

$$\frac{\xi'(s)}{\xi(s)} = \sum_{k=0}^{\infty} \sigma_{k+1}(1-s)^k. \tag{8.10}$$

Taking $s = 1$ in (8.9) gives

$$\sum_{k=1}^{\infty} \sigma_k = -\sigma_1 = b_0 \qquad \text{or} \qquad \sum_{k=2}^{\infty} \sigma_k = 2b_0. \tag{8.11}$$

Theorem 5 gives the relationship between the σ_k and γ_k.

PROOF OF THEOREM 5. It is natural to start from

$$\xi(s) = \frac{s}{2}(s-1)\pi^{-\frac{s}{2}}\Gamma\Big(\frac{s}{2}\Big)\zeta(s). \tag{8.12}$$

Let

$$\ell(s) = (s-1)\zeta(s). \tag{8.13}$$

The logarithmic derivative of (8.12) gives

$$\frac{\xi'(s)}{\xi(s)} = \frac{\ell'(s)}{\ell(s)} - \frac{1}{2}\log\pi + \frac{1}{s} + \frac{1}{2}\psi\Big(\frac{s}{2}\Big). \tag{8.14}$$

Next we consider the Taylor expansion of $\psi\left(\frac{s}{2}\right)$ at $s = 1$. From the duplication formula for $\psi(z)$:

$$\psi(2z) = \frac{1}{2}\Big(\psi(z) + \psi\Big(z + \frac{1}{2}\Big)\Big) + \log 2,$$

we have

$$\begin{aligned} \frac{1}{2}\psi\Big(\frac{s}{2}\Big) &= \psi(s) - \frac{1}{2}\psi\Big(\frac{s+1}{2}\Big) - \log 2 \\ &= -\gamma/2 - \log 2 + \sum_{n=1}^{\infty}(-1)^{n+1}\zeta(n+1)\Big(1 - \frac{1}{2^n}\Big)(s-1)^n \end{aligned} \tag{8.15}$$

(see [1, 6.3.8 and 6.3.14]). Hence from (8.9) and (8.14), we obtain

$$\begin{aligned}\frac{\ell'(s)}{\ell(s)} &= \gamma + \sum_{n=1}^{\infty}(-1)^n\Big\{\zeta(n+1)\Big(1-\frac{1}{2^n}\Big)+\sigma_{n+1}-1\Big\}(s-1)^n \\ &= \gamma + \sum_{n=1}^{\infty}(-1)^n a_n (s-1)^n. \qquad (8.16)\end{aligned}$$

On the other hand,

$$\ell(s) = (s-1)\zeta(s) = 1 - \sum_{n=1}^{\infty}\frac{(-1)^n\gamma_{n-1}}{(n-1)!}(s-1)^n,$$

$$\ell'(s) = \gamma_0 + \sum_{n=1}^{\infty}\frac{(-1)^n(n+1)\gamma_n}{n!}(s-1)^n,$$

so that, from (8.16), we have

$$\gamma_0+\sum_{n=1}^{\infty}\frac{(-1)^n(n+1)\gamma_n}{n!}(s-1)^n = \Big\{1-\sum_{n=1}^{\infty}\frac{(-1)^n\gamma_{n-1}}{n!}(s-1)^n\Big\}\Big\{\gamma+\sum_{n=1}^{\infty}(-1)^n a_n(s-1)^n\Big\}.$$

By comparing the coefficients, we obtain Theorem 5.

REMARK 5. Since

$$\frac{\ell'(s)}{\ell(s)} = \frac{1}{s-1}+\frac{\zeta'(s)}{\zeta(s)},$$

(8.16) can be written in the following form

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1}+\gamma+\sum_{n=1}^{\infty}(-1)^n\Big\{\zeta(n+1)\Big(1-\frac{1}{2^n}\Big)+\sigma_{n+1}-1\Big\}(s-1)^n. \qquad (8.17)$$

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