Characterization of Solvable Quintics $x^5 + ax + b$

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We consider the quintic equation

$$x^5 + ax + b = 0, \tag{1}$$

where a and b are nonzero rational numbers. In general the roots of (1) cannot be expressed as algebraic functions of the coefficients a and b. We will characterize completely those irreducible quintics $x^5 + ax + b$ which are solvable by radicals. We do this by extending Cardano's familiar method of solving the cubic equation $x^3 + ax + b = 0$. We begin by recalling Cardano's method in a way which enables us to apply it to the quintic equation (1).

If u_1, u_2 are complex numbers and ω is a complex cube root of unity, expanding the product

$$\left(x - (u_1 + u_2)\right)\left(x - (\omega u_1 + \omega^2 u_2)\right)\left(x - (\omega^2 u_1 + \omega u_2)\right), \tag{2}$$

we obtain the polynomial

$$x^3 - 3u_1u_2x - \left(u_1^3 + u_2^3\right). \tag{3}$$

As $x_j = \omega^j u_1 + \omega^{2j} u_2$ $(j = 0, 1, 2)$ is a root of the cubic polynomial (2), substituting it into (3), we obtain the identity valid for $j = 0, 1, 2$

$$\left(\omega^j u_1 + \omega^{2j} u_2\right)^3 - 3u_1u_2\left(\omega^j u_1 + \omega^{2j} u_2\right) - \left(u_1^3 + u_2^3\right) = 0.$$

Thus the cubic $x^3 + ax + b = 0$ has the three solutions $x_j = \omega^j u_1 + \omega^{2j} u_2$ $(j = 0, 1, 2)$, where u_1^3 and u_2^3 are determined from $u_1^3 + u_2^3 = -b$, $u_1^3 u_2^3 = -(a/3)^3$.

An obvious generalization of this is to consider the quintic polynomial

$$\prod_{j=0}^{4} \left(x - \left(\omega^j u_1 + \omega^{4j} u_2\right)\right), \tag{4}$$

where ω is now a complex fifth root of unity. Expanding the product (4), and proceeding as above, we find that the quintic $x^5 + ax^3 + (a^2/5)x + b$ (sometimes called DeMoivre's quintic) has the solutions $x_j = \omega^j u_1 + \omega^{4j} u_2$, $j = 0, 1, 2, 3, 4$, where u_1^5 and u_2^5 are determined from $u_1^5 + u_2^5 = -b$, $u_1^5 u_2^5 = -(a/5)^5$.

We refine this method by considering instead of (4) the quintic polynomial

$$\prod_{j=0}^{4} \left(x - \left(\omega^j u_1 + \omega^{2j} u_2 + \omega^{3j} u_3 + \omega^{4j} u_4\right)\right), \tag{5}$$

where u_1, u_2, u_3, u_4 are nonzero real numbers and ω is a complex fifth root of unity. Multiplying out (5) is somewhat more challenging than (4), so MAPLE was employed to do the work. Replacing x by $\omega^j u_1 + \omega^{2j} u_2 + \omega^{3j} u_3 + \omega^{4j} u_4$ in the expanded product, we obtain the identity valid for $j = 0, 1, 2, 3, 4$

$$\begin{aligned}
&\left(\omega^j u_1 + \omega^{2j} u_2 + \omega^{3j} u_3 + \omega^{4j} u_4\right)^5 \\
&\quad -5U\left(\omega^j u_1 + \omega^{2j} u_2 + \omega^{3j} u_3 + \omega^{4j} u_4\right)^3 \\
&\quad\quad -5V\left(\omega^j u_1 + \omega^{2j} u_2 + \omega^{3j} u_3 + \omega^{4j} u_4\right)^2 \\
&\quad\quad +5W\left(\omega^j u_1 + \omega^{2j} u_2 + \omega^{3j} u_3 + \omega^{4j} u_4\right) \\
&\quad\quad\quad +5(X - Y) - Z \\
&\quad\quad\quad = 0,
\end{aligned} \tag{6}$$

where

$$\begin{aligned}
U &= u_1 u_4 + u_2 u_3, \\
V &= u_1 u_2^2 + u_2 u_4^2 + u_3 u_1^2 + u_4 u_3^2, \\
W &= u_1^2 u_4^2 + u_2^2 u_3^2 - u_1^3 u_2 - u_2^3 u_4 - u_3^3 u_1 - u_4^3 u_3 - u_1 u_2 u_3 u_4, \\
X &= u_1^3 u_3 u_4 + u_2^3 u_1 u_3 + u_3^3 u_2 u_4 + u_4^3 u_1 u_2, \\
Y &= u_1 u_3^2 u_4^2 + u_2 u_1^2 u_3^2 + u_3 u_2^2 u_4^2 + u_4 u_1^2 u_2^2, \\
Z &= u_1^5 + u_2^5 + u_3^5 + u_4^5.
\end{aligned}$$

The essential ingredient of the proof of our characterization of solvable quintic trinomials is the determination of real algebraic numbers u_1, u_2, u_3, u_4 satisfying

$$u_1 u_4 + u_2 u_3 = 0, \tag{7}$$

$$u_1 u_2^2 + u_2 u_4^2 + u_3 u_1^2 + u_4 u_3^2 = 0, \tag{8}$$

$$5\left(u_1^2 u_4^2 + u_2^2 u_3^2 - u_1^3 u_2 - u_2^3 u_4 - u_3^3 u_1 - u_4^3 u_3 - u_1 u_2 u_3 u_4\right) = a, \tag{9}$$

and

$$\begin{aligned}
&5\left(\left(u_1^3 u_3 u_4 + u_2^3 u_1 u_3 + u_3^3 u_2 u_4 + u_4^3 u_1 u_2\right)\right. \\
&\quad \left. -\left(u_1 u_3^2 u_4^2 + u_2 u_1^2 u_3^2 + u_3 u_2^2 u_4^2 + u_4 u_1^2 u_2^2\right)\right) \\
&\quad -\left(u_1^5 + u_2^5 + u_3^5 + u_4^5\right) = b,
\end{aligned} \tag{10}$$

so that the quintic polynomial (5) becomes $x^5 + ax + b$ and has the roots

$$x_j = \left(\omega^j u_1 + \omega^{2j} u_2 + \omega^{3j} u_3 + \omega^{4j} u_4\right) \qquad (j = 0, 1, 2, 3, 4). \tag{11}$$

Theorem. *Let a and b be rational numbers such that the quintic trinomial $x^5 + ax + b$ is irreducible. Then the equation $x^5 + ax + b = 0$ is solvable by radicals if and only if there exist rational numbers $\epsilon(= \pm 1)$, $c(\geq 0)$ and $e(\neq 0)$ such that*

$$a = \frac{5e^4(3 - 4\epsilon c)}{c^2 + 1}, \qquad b = \frac{-4e^5(11\epsilon + 2c)}{c^2 + 1}, \tag{12}$$

in which case the roots of $x^5 + ax + b = 0$ are

$$x_j = e\left(\omega^j u_1 + \omega^{2j} u_2 + \omega^{3j} u_3 + \omega^{4j} u_4\right) \qquad (j = 0, 1, 2, 3, 4), \tag{13}$$

where $\omega = \exp(2\pi i/5)$ *and*

$$u_1 = \left(\frac{v_1^2 v_3}{D^2}\right)^{1/5}, \quad u_2 = \left(\frac{v_3^2 v_4}{D^2}\right)^{1/5}, \quad u_3 = \left(\frac{v_2^2 v_1}{D^2}\right)^{1/5}, \quad u_4 = \left(\frac{v_4^2 v_2}{D^2}\right)^{1/5}, \tag{14}$$

$$\begin{cases} v_1 = \sqrt{D} + \sqrt{D - \epsilon\sqrt{D}}, & v_2 = -\sqrt{D} - \sqrt{D + \epsilon\sqrt{D}}, \\ v_3 = -\sqrt{D} + \sqrt{D + \epsilon\sqrt{D}}, & v_4 = \sqrt{D} - \sqrt{D - \epsilon\sqrt{D}}, \end{cases} \tag{15}$$

$$D = c^2 + 1. \tag{16}$$

Proof: We begin by supposing that the irreducible quintic polynomial $x^5 + ax + b$ is solvable by radicals. Thus the resolvent sextic of $x^5 + ax + b$, namely,

$$x^6 + 8ax^5 + 40a^2x^4 + 160a^3x^3 + 400a^4x^2 \\ +(512a^5 - 3125b^4)x + (256a^6 - 9375ab^4)$$

has a rational root r [**1**, Theorem 1]. Hence r satisfies

$$(r + 2a)^4(r^2 + 16a^2) - 5^5b^4(r + 3a) = 0, \tag{17}$$

which shows that $r \neq -2a, -3a$ as $a \neq 0$. We define the nonnegative rational number c and the nonzero rational number e by

$$\epsilon c = \frac{3r - 16a}{4(r + 3a)}, \; e = \frac{-5b\epsilon}{2(r + 2a)}, \quad \text{where } \epsilon = \pm 1. \tag{18}$$

Then

$$c^2 + 1 = \frac{25(r^2 + 16a^2)}{16(r + 3a)^2},$$

$$3 - 4\epsilon c = \frac{25a}{r + 3a},$$

$$11\epsilon + 2c = \frac{25(r + 2a)\epsilon}{2(r + 3a)},$$

so that

$$\frac{5e^4(3 - 4\epsilon c)}{c^2 + 1} = \frac{5^5ab^4(r + 3a)}{(r + 2a)^4(r^2 + 16a^2)} = a,$$

and

$$\frac{-4e^5(11\epsilon + 2c)}{c^2 + 1} = \frac{5^5b^5(r + 3a)}{(r + 2a)^4(r^2 + 16a^2)} = b,$$

giving the required parametrization.

We now show that the irreducible quintic trinomial

$$x^5 + \frac{5e^4(3 - 4\epsilon c)}{c^2 + 1}x - \frac{4e^5(11\epsilon + 2c)}{c^2 + 1} \tag{19}$$

with $e = 1$ is solvable by radicals with roots given by (11). In fact it is not necessary to assume that the quintic is irreducible. For general e the transformation $x \to ex$

gives the required result (13). From (15) we see that

$$\begin{cases} v_1 + v_4 = 2\sqrt{D}, & v_2 + v_3 = -2\sqrt{D}, \\ v_1 v_4 = \epsilon\sqrt{D}, & v_2 v_3 = -\epsilon\sqrt{D}, \end{cases} \tag{20}$$

and so

$$\begin{cases} v_1 + v_2 + v_3 + v_4 = 0, \\ v_1 v_4 + v_2 v_3 = 0. \end{cases} \tag{21}$$

Further, from (14), we obtain

$$u_1^5 = \frac{v_1^2 v_3}{D^2}, \qquad u_2^5 = \frac{v_3^2 v_4}{D^2}, \qquad u_3^5 = \frac{v_2^2 v_1}{D^2}, \qquad u_4^5 = \frac{v_4^2 v_2}{D^2}. \tag{22}$$

Easy calculations making use of (20) and (22) yield

$$u_1 u_4 = -\frac{\epsilon}{\sqrt{D}}, \qquad u_2 u_3 = \frac{\epsilon}{\sqrt{D}}, \tag{23}$$

$$u_1 u_2^2 = \frac{v_3}{D}, \qquad u_3^2 u_4 = \frac{v_2}{D}, \qquad u_1^2 u_3 = \frac{v_1}{D}, \qquad u_4^2 u_2 = \frac{v_4}{D}, \tag{24}$$

and

$$u_1^3 u_2 = \frac{\epsilon v_1 v_3}{D\sqrt{D}}, \qquad u_2^3 u_4 = -\frac{\epsilon v_3 v_4}{D\sqrt{D}}, \qquad u_3^3 u_1 = -\frac{\epsilon v_1 v_2}{D\sqrt{D}}, \qquad u_4^3 u_3 = \frac{\epsilon v_2 v_4}{D\sqrt{D}}, \tag{25}$$

which give the required equations (7) and (8) in view of (21). From (15), (22), (23), (24) and (25), we deduce that

$$\begin{aligned} &5\left(u_1^2 u_4^2 + u_2^2 u_3^2 - u_1^3 u_2 - u_2^3 u_4 - u_3^3 u_1 - u_4^3 u_3 - u_1 u_2 u_3 u_4\right) \\ &\quad = \frac{5\left(3 - 4\epsilon\sqrt{D-1}\right)}{D} = \frac{5(3 - 4\epsilon c)}{c^2 + 1} \end{aligned} \tag{26}$$

and

$$\begin{aligned} &5\big(\left(u_1^3 u_3 u_4 + u_2^3 u_1 u_3 + u_3^3 u_2 u_4 + u_4^3 u_1 u_2\right) \\ &\quad -\left(u_1 u_3^2 u_4^2 + u_2 u_1^2 u_3^2 + u_3 u_2^2 u_4^2 + u_4 u_1^2 u_2^2\right)\big) \\ &\quad -\left(u_1^5 + u_2^5 + u_3^5 + u_4^5\right) = -\frac{\left(44\epsilon + 8\sqrt{D-1}\right)}{D} = -\frac{4(11\epsilon + 2c)}{c^2 + 1}, \end{aligned} \tag{27}$$

which are the required equations (9) and (10). This proves that

$$x^5 + \frac{5(3 - 4\epsilon c)}{c^2 + 1} x - \frac{4(11\epsilon + 2c)}{c^2 + 1}$$

is solvable by radicals and has the roots given in (11).

The discriminant of the trinomial quintic $x^5 + ax + b$ is $4^4 a^5 + 5^5 b^4$ [**2**, p. 259]. The equation $x^5 + ax + b = 0$ has exactly one real root if $4^4 a^5 + 5^5 b^4 > 0$ [**3**, p. 113]. The discriminant of the quintic (19) is

$$\frac{4^4 5^5 e^{20}}{D^5}\left(4\epsilon c^3 - 84c^2 - 37\epsilon c - 122\right)^2 > 0 \tag{28}$$

so that the quintic (19) has exactly one real root. Suppose now that (19) is irreducible over Q. By the Theorem, (19) is solvable by radicals, and so its Galois group is solvable. Hence its Galois group is isomorphic to the Frobenius group F_{20} of order 20, the dihedral group D_5 of order 10, or to the cyclic group of order 5. However (19) has complex roots, so its Galois group cannot be cyclic of order 5. By [**1**, Theorem 2] the Galois group of (19) is the dihedral group D_5 of order 10 if and only if $5D$ is a perfect square in Q. Otherwise the Galois group is the Frobenuis group F_{20} of order 20.

We close with five examples.

Example 1. We consider the quintic $f_1(x) = x^5 - 5x + 12$, which is irreducible as $f_1(x-2)$ is 5-Eisenstein. The resolvent sextic of f_1 is

$$x^6 - 40x^5 + 1000x^4 + 20000x^3 + 250000x^2 - 66400000x + 976000000,$$

which has the rational root $r = 40$. From (18) we see that $\epsilon = 1$, $c = 2$, $e = -1$, so that by (16) $D = 5$. Since $5D = 5^2$ the Galois group of f_1 is D_5. By the Theorem the unique real root of f_1 is

$$\begin{aligned} x = &-\left(\frac{\left(\sqrt{5} + \sqrt{5 - \sqrt{5}}\right)^2\left(-\sqrt{5} + \sqrt{5 + \sqrt{5}}\right)}{25}\right)^{1/5} \\ &-\left(\frac{\left(-\sqrt{5} + \sqrt{5 + \sqrt{5}}\right)^2\left(\sqrt{5} - \sqrt{5 - \sqrt{5}}\right)}{25}\right)^{1/5} \\ &-\left(\frac{\left(-\sqrt{5} - \sqrt{5 + \sqrt{5}}\right)^2\left(\sqrt{5} + \sqrt{5 - \sqrt{5}}\right)}{25}\right)^{1/5} \\ &-\left(\frac{\left(\sqrt{5} - \sqrt{5 - \sqrt{5}}\right)^2\left(-\sqrt{5} - \sqrt{5 + \sqrt{5}}\right)}{25}\right)^{1/5} \end{aligned}$$

A little manipulation shows that this root can be rewritten as

$$x = \tfrac{1}{5}\left(R_1^{1/5} + R_2^{1/5} + R_3^{1/5} + R_4^{1/5}\right),$$

where R_1, R_2, R_3, R_4 are given at the bottom of page 399 of [**1**].

Example 2. We take $f_2(x) = x^5 + 15x + 12$, which is irreducible as $f_2(x)$ is 3-Eisenstein. The resolvent sextic of f_2 is

$$(x + 30)^4(x^2 + 1800) - 2^8 \cdot 3^4 \cdot 5^4(x + 45),$$

which has the rational root $r = 0$. Hence, by (16) and (18), we have $\epsilon = -1$, $c = 4/3$, $e = 1$, $D = 25/9$. Since $5D$ is not the square of a rational number, the Galois group of f_2 is F_{20}. By the Theorem the unique real root of f_2 is

$$\begin{aligned} x = &\left(\frac{-75 - 21\sqrt{10}}{125}\right)^{1/5} + \left(\frac{225 - 72\sqrt{10}}{125}\right)^{1/5} \\ &+\left(\frac{225 + 72\sqrt{10}}{125}\right)^{1/5} + \left(\frac{-75 + 21\sqrt{10}}{125}\right)^{1/5} \end{aligned}$$

in agreement with the more complicated formula given at the top of page 399 in [**1**].

Example 3. Here we take $\epsilon = 1$, $e = 5/2$, $c = 7/24$, so $D = 1 + (\frac{7}{24})^2 = (\frac{25}{24})^2$, and the quintic (19) is $f_3(x) = x^5 + 330x - 4170$, which is irreducible as $f_3(x)$ is 5-Eisenstein. Since $5D = 5^5/(2^6 \cdot 3^2)$ the Galois group of f_3 is F_{20}. By the Theorem the unique real root of f_3 is

$$x = 54^{1/5} + 12^{1/5} + 648^{1/5} - 144^{1/5}.$$

Example 4. Here we take $\epsilon = -1$, $e = 1$, $c = 11/2$, so $D = 125/4$, and the quintic (19) is $f_4(x) = x^5 + 4x$, which is clearly reducible. However, by the remark preceeding (20), the roots of $x^5 + 4x = 0$, namely $x = 0$, $\pm(1 \pm i)$, are given by (13). Here

$$v_1 = \tfrac{1}{2}\left(5\sqrt{5} + \sqrt{5}\sqrt{25 + 2\sqrt{5}}\right), \qquad v_3 = \tfrac{1}{2}\left(-5\sqrt{5} + \sqrt{5}\sqrt{25 - 2\sqrt{5}}\right),$$

$$\begin{aligned}\frac{v_1^2 v_3}{D^2} &= \frac{1}{5^5}\left(1000 - 500\sqrt{5} + 180\sqrt{25 + 2\sqrt{5}} - 240\sqrt{25 - 2\sqrt{5}}\right)\\ &= \frac{1}{5^5}\left(1000 - 500\sqrt{5} + 120\sqrt{5 + 2\sqrt{5}} - 660\sqrt{5 - 2\sqrt{5}}\right),\end{aligned}$$

and

$$u_1 = \left(\frac{v_1^2 v_3}{D^2}\right)^{1/5} = \frac{1}{5}\left(-\sqrt{5} - \sqrt{5 - 2\sqrt{5}}\right).$$

The conjugates of u_1 are

$$\begin{aligned}u_2 &= \tfrac{1}{5}\left(\sqrt{5} - \sqrt{5 + 2\sqrt{5}}\right),\\ u_3 &= \tfrac{1}{5}\left(\sqrt{5} + \sqrt{5 + 2\sqrt{5}}\right),\\ u_4 &= \tfrac{1}{5}\left(-\sqrt{5} + \sqrt{5 - 2\sqrt{5}}\right).\end{aligned}$$

Clearly $x_0 = u_1 + u_2 + u_3 + u_4 = 0$. Further, as

$$\omega = \exp(2\pi i/5) = \left((\sqrt{5} - 1) + i\sqrt{10 + 2\sqrt{5}}\right)/4,$$

we have

$$\begin{aligned}x_1 &= u_1\omega + u_2\omega^2 + u_3\omega^3 + u_4\omega^4\\ &= \tfrac{1}{20}((-x - y)(x - 1 + i(y + z)) + (x - z)(-x - 1 - i(y - z))\\ &\qquad + (x + z)(-x - 1 + i(y - z)) + (-x + y)(x - 1 - i(y + z))),\end{aligned}$$

where $x = \sqrt{5}$, $y = \sqrt{5 - 2\sqrt{5}}$, $z = \sqrt{5 + 2\sqrt{5}}$. Simplifying the expression for x_1, we deduce

$$x_1 = \frac{1}{20}\left(-4x^2 - 2iy^2 - 2iz^2\right) = \frac{-20 - 20i}{20} = -1 - i.$$

We leave it to the reader to show that $x_2 = 1 + i$, $x_3 = 1 - i$, $x_4 = -1 + i$.

Example 5. Let p be a prime with $p \equiv 3 \pmod 4$. We show using the Theorem that the quintic equation $x^5 + 2px + 2p^2 = 0$ is not solvable by radicals. We first observe that $x^5 + 2px + 2p^2$ is 2-Eisenstein so that it is irreducible. Suppose however that the equation is solvable by radicals. Then, by the Theorem, there exist rational numbers $\epsilon(= \pm 1)$, $c(\geq 0)$ and $e(\neq 0)$ such that

$$2p = \frac{5e^4}{c^2+1}(3 - 4\epsilon c), \tag{29}$$

$$2p^2 = -\frac{4e^5}{c^2+1}(11\epsilon + 2c). \tag{30}$$

Expressing the rational numbers c and e in the form $c = m/n$ and $e = r/s$, where m, n, r, s are integers with $\gcd(m, n) = \gcd(r, s) = 1$, and appealing to (29) and (30), we obtain

$$2p(m^2+n^2)s^4 = 5r^4(3n - 4\epsilon m)n, \tag{31}$$

$$2p^2(m^2+n^2)s^5 = -4r^5(11n\epsilon + 2m)n. \tag{32}$$

As p is a prime $\equiv 3 \pmod 4$ and $\gcd(m, n) = 1$, p does not divide $m^2 + n^2$. Further, as $\gcd(r, s) = 1$, it is clear from (31) that p does not divide r. Let $p^\alpha, p^\beta, p^\gamma, p^\delta$ be the exact powers of p dividing n, $3n - 4\epsilon m$, $11\epsilon n + 2m$, s respectively. As p does not divide both of n and $3n - 4\epsilon m$ we see that α or $\beta = 0$. Similarly α or $\gamma = 0$ and β or $\gamma = 0$. Equating powers of p on both sides of (31) and (32), we obtain

$$\begin{cases} 1 + 4\delta = \alpha + \beta, \\ 2 + 5\delta = \alpha + \gamma, \end{cases}$$

which contradicts that at least two of α, β, γ are 0. Hence the equation $x^5 + 2px + 2p^2 = 0$ is not solvable by radicals.

Other examples of solvable quintics are given below together with their Galois groups.

$\epsilon = 1,$	$e = -1,$	$c = 2/11$	$x^5 + 11x + 44$	D_5
$\epsilon = 1,$	$e = -1,$	$c = 0$	$x^5 + 15x + 44$	F_{20}
$\epsilon = -1,$	$e = 1,$	$c = 1/2$	$x^5 + 20x + 32$	D_5
$\epsilon = 1,$	$e = -2,$	$c = 7$	$x^5 - 40x + 64$	F_{20}

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