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SOME INFINITE SERIES INVOLVING THE RIEMANN ZETA FUNCTION

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We derive in a systematic straightforward manner from Euler's infinite product for the reciprocal of the gamma function, the sums of a number of infinite series involving the Riemann zeta function. Our presentation overlaps considerably with, and owes much to, the comprehensive unified treatments of these series by H.M. Srivastava ([9] and [10]). Nevertheless we do obtain some new results. An alternative approach starting with the Euler-Maclaurin summation formula has been given by the first author [14].

1. Families of Infinite Series involving the Riemann Zeta Function

We start our discussion with Euler's infinite product for the reciprocal of the gamma function $\Gamma(z)$, namely,

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}, \tag{1.1}$$

where γ denotes Euler's constant. The formula (1.1) is valid for all complex z and can be found for example in [1, formula 6.1.3, p. 255]. Taking logarithms in

(1.1), we obtain

$$\log \Gamma(z) = -\log z - \gamma z + \sum_{n=1}^{\infty} \left(\frac{z}{n} - \log\left(1 + \frac{z}{n}\right)\right), \tag{1.2}$$

which is valid for all z except $z = -n\,(n = 0, 1, 2, \ldots)$. Differentiating (1.2) we obtain for $z \neq 0, -1, -2, \ldots$

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\frac{1}{z} - \gamma + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+z} \right). \tag{1.3}$$

Replacing z by $1+z$ in (1.3), we deduce for $z \neq -1, -2, \ldots$

$$\psi(1+z) + \gamma = -\frac{1}{1+z} + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1+z} \right) = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+z} \right). \tag{1.4}$$

Equation (1.4) is formula 6.3.16 of [1].

The Riemann zeta function $\zeta(s)$ is an analytic function of s except for a simple pole at $s = 1$ with residue 1. For $\mathrm{Re}(s) > 1$ we have

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}.$$

For $n \geq 2$ we have

$$1 < \zeta(n) \leq \zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} < 2,$$

so that for $|z| < 1$

$$\sum_{n=2}^{\infty} (-1)^n \zeta(n) z^n = \sum_{n=2}^{\infty} (-1)^n z^n \sum_{k=1}^{\infty} \frac{1}{k^n} = \sum_{k=1}^{\infty} \sum_{n=2}^{\infty} \left(-\frac{z}{k} \right)^n = \sum_{k=1}^{\infty} \frac{z^2}{k(k+z)},$$

that is

$$\sum_{n=2}^{\infty} (-1)^n \zeta(n) z^n = z \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+z} \right). \tag{1.5}$$

From (1.4) and (1.5) we obtain

$$\sum_{n=2}^{\infty} (-1)^n \zeta(n) z^n = z\,[\psi(1+z) + \gamma] = 1 + z\,[\psi(z) + \gamma], \quad |z| < 1, \tag{1.6}$$

as $\psi(z+1) = \psi(z) + \frac{1}{z}$ [1, formula 6.3.5, p. 258]. Subtracting

$$\sum_{n=2}^{\infty}(-1)^n z^n = \frac{z^2}{1+z}, \quad |z| < 1,$$

from (1.6) we obtain

$$\sum_{n=2}^{\infty}(-1)^n(\zeta(n)-1)z^n = z\left[\psi(1+z)+\gamma-\frac{z}{1+z}\right], \quad |z| < 1,$$

that is $\left(\text{as } \psi(z+2) = \psi(z+1) + \frac{1}{z+1}\right)$

$$\sum_{n=2}^{\infty}(-1)^n(\zeta(n)-1)z^n = z\left[\psi(z+2)+\gamma-1\right], \quad |z| < 1.$$

Now for $n \geq 2$ we have

$$\begin{aligned}
0 < \zeta(n) - 1 &= \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \cdots \\
&= \frac{1}{2^n}\left(1 + \frac{1}{(3/2)^n} + \frac{1}{(4/2)^n} + \cdots\right) \\
&\leq \frac{1}{2^n}\left(1 + \frac{1}{(3/2)^2} + \frac{1}{(4/2)^2} + \cdots\right) \\
&= \frac{2^2}{2^n}\left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right) \\
&= \frac{2^2}{2^n}\left(\frac{\pi^2}{6} - 1\right) \\
&< \frac{4}{2^n},
\end{aligned}$$

so that $\sum_{n=2}^{\infty}(-1)^n(\zeta(n)-1)z^n$ converges absolutely for $|z| < 2$. But $z[\psi(z+2)+\gamma-1]$ is an analytic function of z for $|z| < 2$, so we have proved the following result (see, for example, Hansen [4, p. 358, formula (54.10.1) with $a = 2$ and $t = -z$]).

Theorem 1. *For* $|z| < 2$

$$\sum_{n=2}^{\infty}(-1)^n[\zeta(n)-1]z^n = z[\psi(z+2)+\gamma-1]. \tag{1.7}$$

Taking particular values of z in (1.7) we obtain the following corollary.

Corollary 1.

$$\sum_{n=2}^{\infty}(-1)^n[\zeta(n)-1]=\frac{1}{2}. \tag{1.8}$$

$$\sum_{n=2}^{\infty}[\zeta(n)-1]=1. \tag{1.9}$$

$$\sum_{n=1}^{\infty}[\zeta(2n)-1]=\frac{3}{4}. \tag{1.10}$$

$$\sum_{n=1}^{\infty}[\zeta(2n+1)-1]=\frac{1}{4}. \tag{1.11}$$

$$\sum_{n=2}^{\infty}(-1)^n[\zeta(n)-1]\left(\frac{3}{2}\right)^n=\frac{31}{10}-3\log 2. \tag{1.12}$$

$$\sum_{n=2}^{\infty}[\zeta(n)-1]\left(\frac{3}{2}\right)^n=3\log 2+\frac{3}{2}. \tag{1.13}$$

$$\sum_{n=1}^{\infty}[\zeta(2n)-1]\left(\frac{9}{4}\right)^n=\frac{23}{10}. \tag{1.14}$$

$$\sum_{n=1}^{\infty}[\zeta(2n+1)-1]\left(\frac{9}{4}\right)^n=2\log 2-\frac{8}{15}. \tag{1.15}$$

Proof. Taking $z=1$ in (1.7), we obtain

$$\sum_{n=2}^{\infty}(-1)^n[\zeta(n)-1]=\psi(3)+\gamma-1.$$

Now

$$\psi(n)=-\gamma+\sum_{k=1}^{n-1}\frac{1}{k}\quad(n\geq 2)\qquad\text{[1, formula 6.3.2, p. 258]}$$

so $\psi(3)=-\gamma+\frac{3}{2}$, which proves (1.8), which is formula (1.7) of [10].

Taking $z=-1$ in (1.7), we obtain

$$\sum_{n=2}^{\infty}[\zeta(n)-1]=1-\gamma-\psi(1),$$

which proves (1.9) as $\psi(1) = -\gamma$ [1, formula 6.3.2, p. 258]. Our formula (1.9) is formula (1.4) of [10].

Adding and subtracting (1.8) and (1.9) yields (1.10) and (1.11). These are given in (1.8) of [10].

Taking $z = \frac{3}{2}$ in (1.7), we have

$$\sum_{n=2}^{\infty}(-1)^n[\zeta(n)-1]\left(\frac{3}{2}\right)^n = \frac{3}{2}\left[\psi\left(\frac{7}{2}\right)+\gamma-1\right].$$

Now [1, formula 6.3.4, p. 258]

$$\psi\left(\frac{2n+1}{2}\right) = -\gamma - 2\log 2 + 2\sum_{k=1}^{n}\frac{1}{2k-1} \quad (n \geq 1)$$

so $\psi\left(\frac{7}{2}\right) = -\gamma - 2\log 2 + \frac{46}{15}$, which proves (1.12).

Taking $z = -\frac{3}{2}$ in (1.7), we have

$$\sum_{n=2}^{\infty}[\zeta(n)-1]\left(\frac{3}{2}\right)^n = -\frac{3}{2}\left[\psi\left(\frac{1}{2}\right)+\gamma-1\right],$$

which proves (1.13) as $\psi\left(\frac{1}{2}\right) = -\gamma - 2\log 2$ [1, formula 6.3.3, p. 258].

Adding and subtracting (1.12) and (1.13) yields (1.14) and (1.15). Formulae (1.12)–(1.15) appear to be new.

Dividing (1.6) and (1.7) by z and integrating, we obtain the following result (see, for example, Srivastava [9, formulae (4.1) and (4.2)]).

Theorem 2.

$$\sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)}{n}z^n = \log\Gamma(1+z)+\gamma z, \quad |z|<1, \tag{1.16}$$

$$\sum_{n=2}^{\infty}(-1)^n\frac{(\zeta(n)-1)}{n}z^n = \log\Gamma(2+z)+(\gamma-1)z, \quad |z|<2. \tag{1.17}$$

Taking particular values of z in Theorem 2, we obtain the formulae of the next corollary.

Corollary 2.

$$\sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)}{n2^n}=\frac{1}{2}(\log\pi+\gamma)-\log 2. \tag{1.18}$$

$$\sum_{n=2}^{\infty}\frac{\zeta(n)}{n2^n}=\frac{1}{2}(\log\pi-\gamma). \tag{1.19}$$

$$\sum_{n=2}^{\infty}(-1)^n\frac{(\zeta(n)-1)}{n}=\log 2+\gamma-1. \tag{1.20}$$

$$\sum_{n=2}^{\infty}\frac{\zeta(n)-1}{n}=1-\gamma. \tag{1.21}$$

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)-1}{n}=\log 2. \tag{1.22}$$

$$\sum_{n=1}^{\infty}\frac{\zeta(2n+1)-1}{2n+1}=1-\gamma-\frac{1}{2}\log 2. \tag{1.23}$$

$$\sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)-1}{n}\left(\frac{3}{2}\right)^n=\log\frac{15}{8}+\frac{1}{2}\log\pi+\frac{3}{2}(\gamma-1). \tag{1.24}$$

$$\sum_{n=2}^{\infty}\frac{\zeta(n)-1}{n}\left(\frac{3}{2}\right)^n=\frac{1}{2}\log\pi+\frac{3}{2}(1-\gamma). \tag{1.25}$$

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)-1}{n}\left(\frac{9}{4}\right)^n=\log\pi+\log\frac{15}{8}. \tag{1.26}$$

$$\sum_{n=1}^{\infty}\frac{\zeta(2n+1)-1}{2n+1}\left(\frac{9}{4}\right)^n=1-\gamma-\frac{1}{3}\log\frac{15}{8}. \tag{1.27}$$

Proof. Taking $z=\frac{1}{2}$ in (1.16), and remembering that $\Gamma(3/2)=\pi^{1/2}/2$ [1, formula 6.1.9, p. 255], we obtain (1.18), which is formula (4.5) of [10].

Next, taking $z=-\frac{1}{2}$ in (1.16), and recalling that $\Gamma(1/2)=\pi^{1/2}$ [1, formula 6.1.8, p. 255], we obtain (1.19), which is formula (4.6) of [10].

With $z=1$ and $z=-1$ in (1.17), we obtain (1.20) and (1.21), respectively, which are (4.4) and (2.33) of [10].

Adding (1.20) and (1.21), we obtain (1.22), which is a result of Johnson [5]. Subtracting (1.20) from (1.21) we deduce (1.23), which is due to Legendre.

Taking $z = \frac{3}{2}$ and $z = -\frac{3}{2}$ in (1.17), we get (1.24) and (1.25) respectively, which are formulae (4.9) and (4.10) of [10].

Finally (1.26) and (1.27) follow by adding and subtracting (1.24) and (1.25).

Integrating (1.7) we have the following theorem (see [10, formula (5.3), p. 13]).

Theorem 3. *For* $|z| < 2$

$$\sum_{n=2}^{\infty}(-1)^n \frac{\zeta(n)-1}{n+1} z^{n+1} = z \log \Gamma(z+2) - \int_0^z \log \Gamma(t+2)\, dt + \frac{1}{2}(\gamma - 1) z^2. \quad (1.28)$$

Corollary 3.

$$\sum_{n=2}^{\infty}(-1)^n \frac{\zeta(n)-1}{n+1} = \frac{1}{2}(3 + \gamma - \log 8\pi). \quad (1.29)$$

$$\sum_{n=2}^{\infty}(-1)^n \frac{\zeta(n)}{n+1} = 1 - \frac{1}{2}(\log 2\pi - \gamma). \quad (1.30)$$

$$\sum_{n=2}^{\infty} \frac{\zeta(n)-1}{n+1} = \frac{1}{2}(3 - \gamma - \log 2\pi). \quad (1.31)$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)-1}{2n+1} = \frac{1}{2}(3 - \log 4\pi). \quad (1.32)$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n+1)-1}{n+1} = \log 2 - \gamma. \quad (1.33)$$

Proof. Taking $z = 1$ in (1.28) we obtain

$$\sum_{n=2}^{\infty}(-1)^n \frac{\zeta(n)-1}{n+1} = \log 2 - \int_0^1 \log \Gamma(t+2)\, dt + \frac{1}{2}(\gamma - 1).$$

But for $q \geq 0$ we have

$$\int_0^1 \log \Gamma(t+q)\, dt = \frac{1}{2} \log 2\pi + q(\log q - 1),$$

see for example [3, formula 6.441(1), p. 661], so that

$$\int_0^1 \log \Gamma(t+2)\,dt = \frac{1}{2}\log 2\pi + 2\log 2 - 2,$$

and (1.29) follows. Formula (1.29) is due to Suryanarayana [11]. Formula (1.30) follows from (1.29) as $\sum_{n=2}^{\infty} \frac{(-1)^n}{n+1} = \log 2 - \frac{1}{2}$.

Next taking $z = -1$ in (1.28) we deduce

$$\sum_{n=2}^{\infty} \frac{\zeta(n)-1}{n+1} = \int_0^{-1} \log \Gamma(t+2)\,dt + \frac{1}{2}(1-\gamma).$$

Now

$$\begin{aligned}\int_0^{-1} \log \Gamma(t+2)\,dt &= -\int_0^1 \log \Gamma(2-t)\,dt \\ &= -\int_0^1 \log \Gamma(1-t)\,dt - \int_0^1 \log(1-t)\,dt \\ &= -\int_0^1 \log \Gamma(t)\,dt - \int_0^1 \log t\,dt \\ &= -\frac{1}{2}\log 2\pi + 1,\end{aligned}$$

so that (1.31) follows. The formulae (1.30) and (1.31) occur in the work of Verma and Kaur [12].

Adding and subtracting (1.29) and (1.31) we obtain (1.32) and (1.33), which are (5.8) and (5.9) of [10].

Replacing z by $-z$ in (1.6) we have

$$\sum_{n=2}^{\infty} \zeta(n) z^n = -z\psi(1-z) - \gamma z, \quad |z| < 1.$$

Adding this to (1.6) we obtain

$$2\sum_{n=2}^{\infty} \zeta(2n) z^{2n} = z\left[\psi(1+z) - \psi(1-z)\right], \quad |z| < 1.$$

From the recurrence formula

$$\psi(1+z) = \psi(z) + \frac{1}{z} \qquad [1, \text{formula 6.3.5, p. 258}]$$

and the reflection formula

$$\psi(1-z) = \psi(z) + \pi \cot \pi z, \qquad \text{[1, formula 6.3.7, p. 259]}$$

we deduce

$$\psi(1+z) - \psi(1-z) = \frac{1}{z} - \pi \cot \pi z,$$

so that

$$\sum_{n=1}^{\infty} \zeta(2n) z^{2n} = \frac{1}{2} - \frac{1}{2}\pi z \cot \pi z, \quad |z| < 1, \tag{1.34}$$

and thus

$$\begin{aligned}\sum_{n=1}^{\infty} \zeta(2n+1) z^{2n+1} &= \sum_{n=2}^{\infty} \zeta(n) z^{n} - \sum_{n=1}^{\infty} \zeta(2n) z^{2n} \\ &= (-z\psi(1-z) - \gamma z) - \left(\frac{1}{2} - \frac{1}{2}\pi z \cot \pi z\right) \\ &= -z\psi(z) - \gamma z - \frac{1}{2} - \frac{1}{2}\pi z \cot \pi z.\end{aligned}$$

We have proved the following well-known result (see, for example, Hansen [4, p. 46, formula (54.3.4)]).

Theorem 4. *For* $|z| < 1$

$$\sum_{n=1}^{\infty} \zeta(2n+1) z^{2n+1} = -\frac{1}{2}(1 + \pi z \cot \pi z) - z(\psi(z) + \gamma). \tag{1.35}$$

From Theorem 4 it is easy to deduce that

$$\sum_{n=1}^{\infty} (\zeta(2n+1) - 1)\, z^{2n+1} = \frac{1}{2} - \frac{1}{2}\pi z \cot \pi z - \frac{z}{1-z^2} + (1-\gamma)z - z\psi(1+z), \quad |z| < 2, \tag{1.36}$$

which is formula 6.3.15 of [1].

The generalized Euler constant $\gamma(r,k)$ is defined for an integer r and a positive integer k by

$$\gamma(r,k) = \lim_{m \to \infty} \left\{ \sum_{\substack{n=1 \\ n \equiv r (\mathrm{mod}\, k)}}^{m} \frac{1}{n} - \frac{1}{k} \log m \right\},$$

so that

$$\gamma(0,1) = \gamma.$$

Lehmer [7, Theorem 7] has shown that

$$\gamma(r,k) = -\{\psi(r/k) + \log k\}/k \qquad (0 < r \le k).$$

Hence

$$\psi(r/k) = -k\gamma(r,k) - \log k \qquad (0 < r \le k). \tag{1.37}$$

Corollary 4. *Let r and k be integers with $0 < r < k$. Then*

$$\sum_{n=1}^{\infty} \zeta(2n+1)\left(\frac{r}{k}\right)^{2n} = \log k - \frac{\pi}{2}\cot(r\pi/k) + k\gamma(r,k) - \gamma - \frac{k}{2r}. \tag{1.38}$$

In particular

$$\sum_{n=1}^{\infty} \frac{\zeta(2n+1)}{2^{2n}} = 2\log 2 - 1. \tag{1.39}$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n+1)}{3^{2n}} = \frac{3}{2}\log 2 - \frac{3}{2}. \tag{1.40}$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n+1)}{4^{2n}} = 3\log 2 - 2. \tag{1.41}$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n+1)}{5^{2n}} = \frac{5}{4}\log 5 + \frac{\sqrt{5}}{2}\log\left(\frac{1+\sqrt{5}}{2}\right) - \frac{5}{2}. \tag{1.42}$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n+1)}{6^{2n}} = 2\log 2 + \frac{3}{2}\log 3 - 3. \tag{1.43}$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n+1)}{8^{2n}} = 4\log 2 + \sqrt{2}\log(1+\sqrt{2}) - 4. \tag{1.44}$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n+1)}{12^{2n}} = (3-\sqrt{3})\log 2 + \frac{3}{2}\log 3 + 2\sqrt{3}\log(1+\sqrt{3}) - 6. \tag{1.45}$$

$$\sum_{n=1}^{\infty} \zeta(2n+1)\left(\frac{2}{3}\right)^{2n} = \frac{3}{2}\log 3 - \frac{3}{4}. \tag{1.46}$$

$$\sum_{n=1}^{\infty} \zeta(2n+1)\left(\frac{3}{4}\right)^{2n} = 3\log 2 - \frac{2}{3}. \tag{1.47}$$

$$\sum_{n=1}^{\infty} \zeta(2n+1)\left(\frac{5}{6}\right)^{2n} = 2\log 2 + \frac{3}{2}\log 3 - \frac{3}{5}. \tag{1.48}$$

Proof. Formula (1.38) follows from (1.35) and (1.37). Formulae (1.39)–(1.48) follow from (1.38) as [7, p. 134]

$$\gamma(1,2) = \frac{1}{2}\gamma + \frac{1}{2}\log 2,$$
$$\gamma(1,3) = \frac{1}{3}\gamma + \frac{\pi}{18}\sqrt{3} + \frac{1}{6}\log 3,$$
$$\gamma(1,4) = \frac{1}{4}\gamma + \frac{\pi}{8} + \frac{1}{4}\log 2,$$
$$\gamma(1,5) = \frac{1}{5}\gamma + \frac{\pi}{10}\sqrt{\frac{5+2\sqrt{5}}{5}} + \frac{1}{20}\log 5 + \frac{\sqrt{5}}{10}\log\left(\frac{1+\sqrt{5}}{2}\right),$$
$$\gamma(1,6) = \frac{1}{6}\gamma + \frac{\pi}{12}\sqrt{3} + \frac{1}{6}\log 2 + \frac{1}{12}\log 3,$$
$$\gamma(1,8) = \frac{1}{8}\gamma + \frac{\pi}{16}(1+\sqrt{2}) + \frac{1}{8}\log 2 + \frac{\sqrt{2}}{8}\log(1+\sqrt{2}),$$
$$\gamma(1,12) = \frac{1}{12}\gamma + \frac{\pi}{24}(2+\sqrt{3}) - \frac{1}{12}(\sqrt{3}-1)\log 2 + \frac{1}{24}\log 3 + \frac{\sqrt{3}}{6}\log(1+\sqrt{3}),$$
$$\gamma(2,3) = \frac{1}{3}\gamma - \frac{\pi}{18}\sqrt{3} + \frac{1}{6}\log 3,$$
$$\gamma(3,4) = \frac{1}{4}\gamma - \frac{\pi}{8} + \frac{1}{4}\log 2,$$
$$\gamma(5,6) = \frac{1}{6}\gamma - \frac{\pi}{12}\sqrt{3} + \frac{1}{6}\log 2 + \frac{1}{12}\log 3.$$

We remark that the values of $\gamma(1,6), \gamma(2,3), \gamma(3,4)$ and $\gamma(5,6)$ are not actually given on p. 134 of [7] but are easily deduced from the expressions given there. For example we have

$$\begin{aligned}
\gamma(1,6) &= \gamma(1,3) - \gamma(4,6) && \text{[7, formula (5), p. 126]}\\
&= \gamma(1,3) - \frac{1}{2}\gamma(2,3) + \frac{1}{6}\log 2 && \text{[7, formula (13), p. 130]}\\
&= \gamma(1,3) - \frac{1}{2}(\gamma - \gamma(0,3) - \gamma(1,3)) + \frac{1}{6}\log 2 && \text{[7, formula (3), p. 126]}\\
&= \frac{1}{6}\log 2 - \frac{1}{2}\gamma + \frac{3}{2}\gamma(1,3) + \frac{1}{2}\left(\frac{\gamma - \log 3}{3}\right) && \text{[7, formula (2), p. 126]}\\
&= \frac{1}{6}\log 2 - \frac{1}{6}\log 3 - \frac{1}{3}\gamma + \frac{3}{2}\left(\frac{1}{3}\gamma + \frac{\pi}{18}\sqrt{3} + \frac{1}{6}\log 3\right) && \text{[7, p. 134]}\\
&= \frac{1}{6}\gamma + \frac{\pi}{12}\sqrt{3} + \frac{1}{6}\log 2 + \frac{1}{12}\log 3,
\end{aligned}$$

as asserted.

In a similar manner we can determine $\sum_{n=1}^{\infty} \zeta(2n+1)\alpha^{2n}$ explicitly for $\alpha = \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12}$. Making use of (1.36) we can find the sums of such infinite series as $\sum_{n=1}^{\infty} (\zeta(2n+1)-1)\left(\frac{9}{4}\right)^n$.

Rewriting (1.35) as

$$\sum_{n=1}^{\infty} \zeta(2n+1)z^{2n} = \frac{1}{2z} - \frac{\pi}{2}\cot \pi z - \psi(1+z) - \gamma, \quad |z| < 1, \tag{1.49}$$

and integrating, we obtain the following well-known result (see, for example, Hansen [4, p. 356, formula (54.5.8)]).

Theorem 5. *For* $|z| < 1$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n+1)}{2n+1} z^{2n+1} = \frac{1}{2}\log \frac{\pi z}{\sin \pi z} - \log \Gamma(1+z) - \gamma z. \tag{1.50}$$

Appealing to the recurrence and reflection formulae for the gamma function [1, formulae 6.1.15 and 6.1.17, p. 256], we obtain formula (4.12) of [10]. Taking $z = 1/2$ in (1.50) we obtain the following formula, which was obtained by Euler in 1781.

Corollary 5.

$$\sum_{n=1}^{\infty} \frac{\zeta(2n+1)}{(2n+1)2^{2n}} = \log 2 - \gamma. \tag{1.51}$$

Differentiating (1.49) we have the next theorem.

Theorem 6. *For* $|z| < 1$

$$\sum_{n=1}^{\infty} 2n\zeta(2n+1)z^{2n-1} = \frac{1}{2}\left(\frac{\pi^2}{\sin^2 \pi z} + \frac{1}{z^2}\right) - \psi'(z), \tag{1.52}$$

$$\sum_{n=1}^{\infty} (2n+1)\zeta(2n+1)z^{2n} = \frac{1}{2}\left(\frac{\pi^2 z}{\sin^2 \pi z} - \pi \cot \pi z\right) - \psi(z) - \gamma - \psi'(z)z. \tag{1.53}$$

Corollary 6.

$$\sum_{n=1}^{\infty} \frac{2n\zeta(2n+1)}{2^{2n}} = 1. \tag{1.54}$$

$$\sum_{n=1}^{\infty} \frac{(2n+1)\zeta(2n+1)}{2^{2n+1}} = \log 2. \tag{1.55}$$

Proof. Taking $z = 1/2$ in (1.52) we have

$$\sum_{n=1}^{\infty} \frac{2n\zeta(2n+1)}{2^{2n-1}} = \frac{\pi^2}{2} + 2 - \psi'\left(\frac{1}{2}\right).$$

Formula (1.54) now follows as $\psi'\left(\frac{1}{2}\right) = \frac{\pi^2}{2}$ [1, formula 6.4.4, p. 260].

Taking $z = 1/2$ in (1.53) we deduce

$$\sum_{n=1}^{\infty} \frac{(2n+1)\zeta(2n+1)}{2^{2n+1}} = \frac{\pi^2}{8} - \frac{1}{2}\left(\psi\left(\frac{1}{2}\right) + \gamma\right) - \frac{1}{2}\psi'\left(\frac{1}{2}\right) = \log 2.$$

The formulae (1.54) and (1.55) appear to be new.

Changing z into iz and recalling that

$$\cot \pi i z = \frac{1}{i}\coth \pi z, \qquad \zeta(0) = -\frac{1}{2},$$

formula (1.34) becomes

$$\pi z \coth \pi z = -2\sum_{n=0}^{\infty}(-1)^n \zeta(2n) z^{2n}, \quad |z| < 1.$$

Manipulating this formula we get the next theorem (see, for example, Hansen [4, p. 356, formula (54.3.3)]).

Theorem 7. *For* $|z| < 1$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\zeta(2n)z^{2n-1} = \frac{\pi}{2}\coth \pi z - \frac{1}{2z}, \tag{1.56}$$

and for $|z| \le 1$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\left(\zeta(2n)-1\right)z^{2n-1} = \frac{\pi}{2}\coth \pi z - \frac{(3z^2+1)}{2z(z^2+1)}. \qquad (1.57)$$

Corollary 7.

$$\sum_{n=1}^{\infty}(-1)^{n-1}\left(\zeta(2n)-1\right) = \frac{\pi}{2}\coth \pi - 1. \qquad (1.58)$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\zeta(2n)}{2^{2n}} = \frac{\pi}{4}\coth\frac{\pi}{2} - \frac{1}{2}. \qquad (1.59)$$

Proof. Formulae (1.58) and (1.59) result by taking $z = 1$ in (1.57) and $z = 1/2$ in (1.56).

The formulae of Corollary 7 are formulae (15) and (16) of [14] suitably corrected.

Integrating (1.56) we obtain (see, for example, Hansen [4, p. 356, formula (54.5.3) with $t = iz$]).

Theorem 8. *For* $|z| \le 1$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\zeta(2n)}{n}z^{2n} = \log\left(\frac{\sinh \pi z}{\pi z}\right). \qquad (1.60)$$

Taking $z = 1$ and $z = 1/2$ in (1.60) we obtain

Corollary 8.

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\zeta(2n)}{n} = \log\left(\frac{\sinh \pi}{\pi}\right). \qquad (1.61)$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\zeta(2n)}{n2^{2n}} = \log\left(\frac{2\sinh \pi/2}{\pi}\right). \qquad (1.62)$$

A variant of formula (1.61) can be found in Knopp [6, p. 271, Exercise 124(f)].

We conclude this section by remarking that many of the formulae of the corollaries can be obtained by changing the order of summation and then appealing as necessary to known results. We give three examples.

Proof of (1.20). We have

$$\begin{aligned}\sum_{n=2}^{\infty}(-1)^{n-1}\frac{\zeta(n)}{n} &= \sum_{n=2}^{\infty}\frac{(-1)^n}{n}\sum_{k=1}^{\infty}\frac{1}{k^n}\\ &= \sum_{k=1}^{\infty}\sum_{n=2}^{\infty}\frac{\left(-\frac{1}{k}\right)^n}{n}\\ &= \sum_{k=1}^{\infty}\left(\frac{1}{k}-\log\left(1+\frac{1}{k}\right)\right)\\ &= \gamma,\end{aligned}$$

from which (1.20) follows.

Proof of (1.58). We have

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n-1}\left(\zeta(2n)-1\right) &= \sum_{n=1}^{\infty}(-1)^{n-1}\sum_{k=2}^{\infty}\frac{1}{k^{2n}}\\ &= \sum_{k=2}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{k^{2n}}\\ &= \sum_{k=2}^{\infty}\frac{1}{1+k^2}\\ &= \sum_{k=1}^{\infty}\frac{1}{1+k^2}-\frac{1}{2}\\ &= \frac{\pi}{2}\coth\pi-1,\end{aligned}$$

since [3, formula 1.445]

$$\sum_{k=1}^{\infty}\frac{1}{k^2+a^2}=\frac{\pi}{2a}\coth\pi a-\frac{1}{2a^2}\qquad(a\neq 0).$$

Proof of (1.61).

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\zeta(2n)}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sum_{k=1}^{\infty} \frac{1}{k^{2n}}$$
$$= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{1}{k^2}\right)^n$$
$$= \sum_{k=1}^{\infty} \log\left(1+\frac{1}{k^2}\right)$$
$$= \log \prod_{k=1}^{\infty} \left(1+\frac{1}{k^2}\right)$$
$$= \log \frac{\sinh \pi}{\pi},$$

since [3, formula 1.431, p. 37]

$$\sinh \pi x = \pi x \prod_{k=1}^{\infty} \left(1+\frac{x^2}{k^2}\right).$$

Many other formulae can be derived from those above by taking linear combinations: for example $\frac{1}{2}$ (1.22)-(1.32) gives (as $\sum_{n=1}^{\infty} \frac{1}{n(2n+1)} = 2 - 2\log 2$)

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)} = \log 2\pi - 1,$$

a result which is due to Wilton [13].

2. Three Infinite Series Representations of $\zeta(3)$

We may rewrite (1.34) in the form

$$\frac{1}{z} - \cot z = 2\sum_{n=1}^{\infty} \frac{\zeta(2n)}{\pi^{2n}} z^{2n-1}, \quad |z| < \pi. \tag{2.1}$$

Integrating (2.1) yields

$$\log z - \log \sin z = 2\sum_{n=1}^{\infty} \frac{\zeta(2n)}{2n\pi^{2n}} z^{2n}, \quad |z| < \pi. \tag{2.2}$$

Replacing z by $z/2$ in (2.2) gives

$$\log \frac{z}{2\sin z/2} = \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2\pi)^{2n}} z^{2n}, \quad |z| < 2\pi.$$

Integrating from $z = 0$ to $z = x$ we deduce

$$x\log x - x - \int_0^x \log\left(2\sin\frac{z}{2}\right) dz = \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)(2\pi)^{2n}} x^{2n+1}, \quad |x| < 2\pi.$$

Integrating again yields for $|x| < 2\pi$

$$\frac{x^2}{2}\log x - \frac{3x^2}{4} - \int_0^x \left(\int_0^u \log\left(2\sin\frac{t}{2}\right) dt\right) du = \sum_{n=1}^{\infty} \frac{\zeta(2n)x^{2n+2}}{n(2n+1)(2n+2)(2\pi)^{2n}}.$$

Taking $x = \pi/3$ gives

$$\frac{\pi^2}{18}\log\frac{\pi}{3} - \frac{\pi^2}{12} - \int_0^{\pi/3} \left(\int_0^u \log\left(2\sin\frac{t}{2}\right) dt\right) du = \frac{\pi^2}{9}\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)(2n+2)6^{2n}}.$$

Integrating by parts we deduce

$$\frac{\pi^2}{18}\log\frac{\pi}{3} - \frac{\pi^2}{12} + \int_0^{\pi/3} \left(x - \frac{\pi}{3}\right) \log\left(2\sin\frac{x}{2}\right) dx = \frac{\pi^2}{9}\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)(2n+2)6^{2n}}.$$

Since [8, p. 230]

$$\zeta(3) = \frac{3}{2}\int_0^{\pi/3} \left(x - \frac{\pi}{3}\right) \log\left(2\sin\frac{x}{2}\right) dx,$$

we obtain our first formula for $\zeta(3)$.

Theorem 9.

$$\zeta(3) = \frac{\pi^2}{8} - \frac{\pi^2}{12}\log\frac{\pi}{3} + \frac{\pi^2}{6}\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)(2n+2)6^{2n}}.$$

Next, recalling that $\zeta(0) = -\frac{1}{2}$, we can rewrite (2.1) in the form

$$-u\cot u = 2\sum_{n=0}^{\infty} \frac{\zeta(2n)}{\pi^{2n}} u^{2n}, \quad |u| < \pi. \tag{2.3}$$

Integrating (2.3) between 0 and t, we obtain

$$-\int_0^t u\cot u\,du = 2\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+1)\pi^{2n}}t^{2n+1}, \quad |t|<\pi. \tag{2.4}$$

Now integrating (2.4) between 0 and $\pi/2$ we have

$$-\int_0^{\pi/2}\left(\int_0^t u\cot u\,du\right)dt = \frac{\pi^2}{2}\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+1)(2n+2)2^{2n}}.$$

Changing the order of integration, we obtain

$$-\frac{\pi}{2}\int_0^{\pi/2} u\cot u\,du + \int_0^{\pi/2} u^2\cot u\,du = \frac{\pi^2}{2}\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+1)(2n+2)2^{2n}}.$$

Then, integrating by parts, we deduce

$$\frac{\pi}{2}\int_0^{\pi/2}\log\sin u\,du - 2\int_0^{\pi/2} u\log\sin u\,du = \frac{\pi^2}{2}\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+1)(2n+2)2^{2n}}.$$

Recalling that

$$\int_0^{\pi/2}\log\sin x\,dx = -\frac{\pi}{2}\log 2 \qquad \text{[1, formula 4.3.145, p. 78]} \tag{2.5}$$

and

$$\int_0^{\pi} x\log\left(2\sin\frac{x}{2}\right)dx = \frac{7}{4}\zeta(3) \qquad \text{[8, p. 164]} \tag{2.6}$$

we have our second series for $\zeta(3)$.

Theorem 10.

$$\zeta(3) = -\frac{4\pi^2}{7}\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+1)(2n+2)2^{2n}}.$$

This formula can be found for example in [10, formula (2.23), p. 7], [2, formula (1), p. 219] and [15, formula (1.2)].

Finally, multiplying (2.3) by u, integrating twice and taking $x=\pi/2$, we have

$$-\int_0^{\pi/2}\left(\int_0^t u^2\cot u\,du\right)dt = \frac{\pi^3}{4}\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+2)(2n+3)2^{2n}}.$$

Interchanging the order of integration, we obtain

$$-\frac{\pi}{2}\int_0^{\pi/2} u^2 \cot u du + \int_0^{\pi/2} u^3 \cot u du = \frac{\pi^3}{4}\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+1)(2n+3)2^{2n}}.$$

Integrating by parts yields

$$\pi\int_0^{\pi/2} u \log\sin u du - 3\int_0^{\pi/2} u^2 \log\sin u du = \frac{\pi^3}{4}\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+2)(2n+3)2^{2n}},$$

that is

$$\pi^3\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+2)(2n+3)2^{2n}} = -\int_0^{\pi/2}(12u^2 - 4\pi u)\log\sin u du.$$

Now

$$\begin{aligned}
\zeta(3) &= \frac{(2\pi)^3}{2\cdot 3!}\int_0^1 B_3(x)\cot\pi x dx \qquad [1, \text{formula } 23.2.17, \text{ p. } 807] \\
&= \frac{(2\pi)^3}{3!}\int_0^{1/2} B_3(x)\cot\pi x dx \qquad [1, \text{formula } 23.1.8, \text{ p. } 804] \\
&= \frac{(2\pi)^3}{3!\pi}\int_0^{1/2} B_3(x) d(\log\sin\pi x) \\
&= -\frac{4\pi^2}{3}\int_0^{1/2} B_3'(x)\log\sin\pi x dx \\
&= -4\pi^2\int_0^{1/2} B_2(x)\log\sin\pi x dx \qquad [1, \text{formula } 23.1.5, \text{ p. } 804] \\
&= -4\pi^2\int_0^{1/2}(x^2 - x + \frac{1}{6})\log\sin\pi x dx \\
&= -\frac{4}{\pi}\int_0^{\pi/2}\left(u^2 - \pi u + \frac{\pi^2}{6}\right)\log\sin u du
\end{aligned}$$

and

$$\int_0^{\pi/2}\left(2u-\frac{\pi}{2}\right)\log\sin u\,du$$
$$=\frac{1}{2}\int_0^{\pi}\left(x-\frac{\pi}{2}\right)\log\sin\frac{x}{2}dx$$
$$=\frac{1}{2}\int_0^{\pi}x\log\sin\frac{x}{2}dx-\frac{\pi}{4}\int_0^{\pi}\log\sin\frac{x}{2}dx$$
$$=\frac{1}{2}\int_0^{\pi}x\log\left(2\sin\frac{x}{2}\right)dx-\frac{\log 2}{2}\int_0^{\pi}x\,dx-\frac{\pi}{2}\int_0^{\pi/2}\log\sin y\,dy$$
$$=\frac{7}{8}\zeta(3)-\frac{\pi^2}{4}\log 2+\frac{\pi^2}{4}\log 2 \qquad \text{(by (2.5) and (2.6))}$$
$$=\frac{7}{8}\zeta(3),$$

so that

$$\pi^3\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+2)(2n+3)2^{2n}}$$
$$=\int_0^{\pi/2}\left(-12\left(u^2-\pi u+\frac{\pi^2}{6}\right)-4\pi\left(2u-\frac{\pi}{2}\right)\right)\log\sin u\,du$$
$$=3\pi\zeta(3)-\frac{7}{2}\pi\zeta(3)=-\frac{\pi}{2}\zeta(3).$$

We have proved the following result.

Theorem 11.

$$\zeta(3)=-2\pi^2\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+2)(2n+3)2^{2n}}.$$

The formula of Theorem 11 was obtained in a different way in [15].

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