THE REPRESENTATION OF A PAIR OF INTEGERS
BY A PAIR OF POSITIVE-DEFINITE BINARY QUADRATIC FORMS

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ABSTRACT. An explicit formula is given for the number of representations of a
pair of positive integers by a representative set of inequivalent pairs of integral
positive-definite binary quadratic forms with given invariants.

0. Notation

By a form we mean a binary quadratic form \( f = (a, b, c) = aX^2 + bXY + cY^2 \),
which is integral (that is \( a, b, c \) are integers), positive definite (that is \( a > 0 \),
\( b^2 - 4ac < 0 \)) and primitive (that is \( \text{GCD}(a, b, c) = 1 \)). The discriminant of
\( f \), written \( \text{disc}(f) \), is the integer \( b^2 - 4ac \).

1. Introduction

Two forms \( f \) and \( f' \) are said to be equivalent (written \( f \sim f' \)) if there
exists a transformation

\[
\tau: \begin{pmatrix} X \\ Y \end{pmatrix} \rightarrow \begin{pmatrix} r & s \\ t & u \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix},
\]

where \( r, s, t, u \) are integers satisfying \( ru - st = 1 \), such that

\[
f(rX + sY, tX + uY) = f'(X, Y).
\]

The transformation \( \tau \) preserves \( \text{disc}(f) \). The relation \( \sim \) is an equivalence
relation on the set of forms with given discriminant \( d \). It is well known that
the number \( h(d) \) of equivalence classes is finite. Let

\[
f_i = a_iX^2 + b_iXY + c_iY^2, \quad i = 1, 2, \ldots, h(d),
\]

be a representative set of inequivalent forms of discriminant \( d \). The positive

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847
integer $m$ is said to be represented by the form $f_i$ if there exist integers $x$ and $y$ such that

$$m = f_i(x, y).$$

The number of pairs $(x, y)$ of integers satisfying (1.4) is denoted by $\psi_d^{(i)}(m)$. Clearly $\psi_d^{(i)}(m)$ is unchanged if the form $f_i$ is replaced by another form equivalent to it. The total number of representations of $m$ by a representative set of inequivalent forms of discriminant $d$ is

$$\psi_d(m) = \sum_{i=1}^{h(d)} \psi_d^{(i)}(m).$$

In [1] Dirichlet proved that if $\text{GCD}(m, 2d) = 1$ then

$$\psi_d(m) = w(d) \sum_{e|m} \left( \frac{d}{e} \right),$$

where $e$ runs through all the positive integers dividing $m$, $(d/e)$ is the Kronecker symbol and

$$w(d) = \begin{cases} 
4, & \text{if } d = -4, \\
6, & \text{if } d = -3, \\
2, & \text{if } d \neq -3, -4.
\end{cases}$$

In this paper we consider the representability of a pair of positive integers $(m, M)$ by pairs of forms and obtain results analogous to Dirichlet’s formula (1.6).

2. Pairs of forms

Two pairs of forms $(f, F) = (ax^2 + bxy + cy^2, Ax^2 + Bxy + Cy^2)$ and $(f', F')$ are said to be equivalent, written $(f, F) \sim (f', F')$, if there exists a transformation $\tau$ of the type given in (1.1) such that

$$(f(rX + sY, tX + uY), F(rX + sY, tX + uY)) = (f'(X, Y), F'(X, Y)).$$

The transformation $\tau$ preserves $d = \text{disc}(f) = b^2 - 4ac$, $D = \text{disc}(F) = B^2 - 4AC$, as well as the codiscriminant $\Delta = \text{codisc}(f, F) = bB - 2aC - 2cA$ of the pair $(f, F)$ [3]. From now on we suppose that $d$, $D$, and $\Delta$ are given and that there are pairs of forms $(f, F)$ with $\text{disc}(f) = d$, $\text{disc}(F) = D$, and $\text{codisc}(f, F) = \Delta$. It is easy to prove [2] that

$$\Delta < 0, \quad \Delta^2 - dD \geq 0.$$ 

If $\Delta^2 - dD = 0$ it is straightforward [2] to show that $d = D = \Delta$ and that any pair $(f, F)$ with these invariants must have $f = F$. Thus in this case equivalence of pairs of forms reduces to the equivalence of forms described in §1. Thus we may exclude this case and assume from now on that

$$\Delta^2 - dD > 0.$$
On the set of pairs of forms \((f_i, F_i)\) with specified \(d, D\), and \(\Delta\), the relation \(\sim\) is an equivalence relation, and the number \(h(d, D, \Delta)\) of equivalence classes is finite [3]. A formula for \(h(d, D, \Delta)\) has been given by Hardy and Williams [2] in the case when \(d\) and \(D\) are fundamental discriminants and \(GCD(dD, \Delta) = 2^l\) for some \(l \geq 0\). We let

\[
(2.4) \quad (f_i, F_i) = (a_iX^2 + b_iXY + c_iY^2, A_iX^2 + B_iXY + C_iY^2), \quad i = 1, 2, \ldots, h(d, D, \Delta),
\]

be a representative set of inequivalent pairs of forms with given \(d, D, \Delta\). We say that the pair \((m, M)\) of positive integers is represented by the pair \((f_i, F_i)\) if there exist integers \(x, y\) such that

\[
(2.5) \quad m = f_i(x, y), \quad M = F_i(x, y).
\]

The number of pairs of integers \((x, y)\) satisfying (2.5) is denoted by \(\Psi_{d, D, \Delta}^{(i)}(m, M)\). Clearly \(\Psi_{d, D, \Delta}^{(i)}(m, M)\) is unaltered if the pair \((f_i, F_i)\) is replaced by another pair of forms equivalent to \((f_i, F_i)\). The total number of representations of \((m, M)\) by a representative set of inequivalent pairs of forms is

\[
(2.6) \quad \Psi_{d, D, \Delta}(m, M) = \sum_{i=1}^{h(d, D, \Delta)} \Psi_{d, D, \Delta}^{(i)}(m, M).
\]

We prove the following theorem which gives the value of \(\Psi_{d, D, \Delta}(m, M)\) for all positive integers \(m, M\) for which

\[
(2.7) \quad GCD(m, 2d(\Delta^2 - dD)) = GCD(M, 2D(\Delta^2 - dD)) = 1.
\]

**Theorem.** (a) If \(dM^2 - 2\Delta Mm + Dm^2\) is not a square then

\[
(2.8) \quad \Psi_{d, D, \Delta}(m, M) = 0.
\]

(b) If \(dM^2 - 2\Delta Mm + Dm^2 = k^2\) for some integer \(k\) and

\[
(2.9) \quad GCD(m, M) = GCD(m, 2d) = GCD(M, 2D) = 1
\]

then

\[
(2.10) \quad \Psi_{d, D, \Delta}(m, M) = \begin{cases} 4, & \text{if } k \neq 0, \\ 2, & \text{if } k = 0. \end{cases}
\]

(c) If \(dM^2 - 2\Delta Mm + Dm^2 = k^2\) for some integer \(k\) and

\[
(2.11) \quad GCD(m, 2d(\Delta^2 - dD)) = GCD(M, 2D(\Delta^2 - dD)) = 1
\]

then

\[
(2.12) \quad \Psi_{d, D, \Delta}(m, M) = \begin{cases} 4, & \text{if } k \neq 0 \text{ and } GCD(m, M) = l^2 \text{ for some integer } l, \\ 2, & \text{if } k = 0 \text{ and } GCD(m, M) = l^2 \text{ for some integer } l, \\ 0, & \text{if } GCD(m, M) \neq l^2 \text{ for any integer } l. \end{cases}
\]
3. Proof of theorem (a)

If \( \Psi_{d, D, \Delta}(m, M) \geq 1 \) then there are integers \( x \) and \( y \) and an integer \( i (1 \leq i \leq h(d, D, \Delta)) \) such that

\[
\begin{align*}
m &= a_i x^2 + b_i x y + c_i y^2 \\
M &= A_i x^2 + B_i x y + C_i y^2,
\end{align*}
\]

and so

\[
dM^2 - 2\Delta Mm + Dm^2 = k^2,
\]

where

\[
\pm k = (a_i B_i - b_i A_i) x^2 + 2(a_i C_i - c_i A_i) x y + (b_i C_i - c_i B_i) y^2.
\]

Hence if \( dM^2 - 2\Delta Mm + Dm^2 \) is not a square, we must have \( \Psi_{d, D, \Delta}(m, M) = 0 \).

4. Proof of theorem (b)

Throughout this section we assume that \( m, M \) are positive integers satisfying (2.9) and that there exists an integer \( k \) such that (3.2) holds. The number of pairs of integers \( n \pmod{2m} \) and \( N \pmod{2M} \) such that

\[
n^2 \equiv d \pmod{4m}, \quad N^2 \equiv D \pmod{4M},
\]

and for which

\[
\text{there exist representatives satisfying } Mn - mN = k,
\]

is denoted by \( A(m, M) \). We begin by determining \( A(m, M) \).

**Lemma 1.** \( A(m, M) = 1 \).

**Proof.** Clearly, for any solution of (4.1) satisfying (4.2), one has

\[
Mn \equiv k \pmod{m}, \quad mN \equiv -k \pmod{M}.
\]

Conversely, for any pair of integers \( (n_0, N_0) \) for which (4.1) and (4.3) hold, we have

\[
Mn_0 - mN_0 \equiv k \pmod{m},
\]

\[
Mn_0 - mN_0 \equiv k \pmod{M},
\]

\[
Mn_0 - mN_0 \equiv M^2 n_0^2 + m^2 N_0^2 \equiv dM^2 + Dm^2 \equiv k^2 \equiv k \pmod{2} \text{ (by (3.2))},
\]

and so

\[
Mn_0 - mN_0 \equiv k \pmod{2mM}.
\]

Noting that

\[
M(n_0 + 2mr) - (N_0 + 2MR) = (Mn_0 - mN_0) + 2mM(r - R),
\]

we see that the classes of \( n_0 \pmod{2m} \) and \( N_0 \pmod{2M} \) contain representatives \( n \) and \( N \) satisfying \( Mn - mN = k \), that is (4.2) holds. Thus we have

\[
A(m, M) = B(d, m, M, k)B(D, M, m, -k),
\]
where \( B(d, m, M, k) \) is the number of solutions \( n \pmod{2m} \) of

\[
(4.5) \quad n^2 \equiv d \pmod{4m}, \quad Mn \equiv k \pmod{m}.
\]

The congruence \( Mn \equiv k \pmod{m} \) has a unique solution \( n_0 \pmod{m} \). For this solution the congruence \( n_0^2 \equiv d \pmod{m} \) is automatically true in view of (3.2). The solutions \( \pmod{2m} \) of \( Mn \equiv k \pmod{m} \) are given by

\[
n_0 + \varepsilon m, \quad \varepsilon = 0 \text{ or } 1.
\]

These solutions satisfy \( n^2 \equiv d \pmod{4} \) for the unique value of \( \varepsilon \) such that

\[
(n_0 + \varepsilon)^2 \equiv d \pmod{4}.
\]

Thus we have \( B(d, m, M, k) = 1 \) and similarly \( B(D, M, m, -k) = 1 \). Hence (4.4) gives \( A(m, M) = 1 \) as required. \( \square \)

The next lemma gives the automorphs of a pair of forms \( (f, F) \).

**Lemma 2.** The only transformations

\[
\tau \colon \begin{pmatrix} X \\ Y \end{pmatrix} \rightarrow \begin{pmatrix} r & s \\ t & u \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}, \quad (ru - st = 1)
\]

mapping the pair of forms \( (f, F) \) into itself are given by

\[
\begin{pmatrix} r & s \\ t & u \end{pmatrix} = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

**Proof.** If \( d \neq -3, -4 \) the only automorphs of the form \( f = ax^2 + bxy + cy^2 \) of discriminant \( d \) are

\[
\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Thus the assertion of the lemma is clear unless \( (d, D) = (-3, -3), (-3, -4), (-4, -3) \) or \( (-4, -4) \).

We just treat the case \( (d, D) = (-3, -3) \) as the other cases can be treated similarly. As every form of discriminant \( -3 \) is equivalent to the form \( (1, 1, 1) \) we may suppose by applying a suitable transformation to \( f \) that \( f = (1, 1, 1) \). The only automorphs of \( f \) are

\[
\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \pm \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad \text{and} \quad \pm \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.
\]

The second of these transforms \( F = (A, B, C) \) into \( (C, -B + 2C, A - B + C) \) and so can only be an automorph for the pair \( (f, F) \) if \( A = C, B = -B + 2C, C = A - B + C \), that is \( A = B = C \), i.e., \( F = (1, 1, 1) \), and thus \( d = D = \Delta = -3 \) which is impossible as \( \Delta^2 - dD \neq 0 \). The third mapping transforms \( F = (A, B, C) \) into \( (A - B + C, 2A - B, A) \), and, exactly as above, we see that it cannot be an automorph of the pair \( (f, F) \). This completes the proof of Lemma 2. \( \square \)

The next lemma is easily checked.
Lemma 3. If \( d = n^2 - 4ml \), \( D = N^2 - 4ML \) then the following is an identity
\[
dM^2 + Dm^2 - (mN - Mn)^2 = 2mM(nN - 2mL - 2ML).
\]

We are now ready to prove Theorem (b). If \((x, y)\) is a pair of integers, we set
\[
[x, y] = \{(x, y), (-x, -y)\}
\]
and for \( i = 1, 2, \ldots, h(D, D, \Delta) \) we let
\[
S_i = \{[x, y] | m = a_i x^2 + b_i xy + c_i y^2, \ M = A_i x^2 + B_i xy + C_i y^2\}.
\]
We remark that if \([x, y] \in S_i\) then \(GCD(x, y) = 1\) as \(GCD(m, M) = 1\). The set of all pairs \(([x, y], i)\) with \([x, y] \in S_i\) and \(i = 1, 2, \ldots, h(D, D, \Delta)\) is denoted by \(S\). Clearly we have
\[
(4.6) \quad \text{card}(S) = \frac{1}{2} \Psi_{d, D, \Delta}(m, M).
\]
Recalling that \(m\) and \(M\) are positive integers satisfying (2.9) and for which \(dM^2 - 2\Delta Mm + Dm^2 = k^2\) is solvable, we set
\[
C_{m, M} = \{(n(\text{mod } 2m), N(\text{mod } 2M)) | n^2 \equiv d(\text{mod } 4m), N^2 \equiv D(\text{mod } 4M),
\]
\[
Mn - mN = \pm k\}.
\]
By Lemma 1 we have
\[
(4.7) \quad \text{card}(C_{m, M}) = \begin{cases} 2, & \text{if } k \neq 0, \\ 1, & \text{if } k = 0. \end{cases}
\]

Next we define a mapping \(T: S \to C_{m, M}\) as follows: if \([x, y] \in S_i\), where \(1 \leq i \leq h(D, D, \Delta)\), then
\[
T(([x, y], i)) = (n(\text{mod } 2m), N(\text{mod } 2M)),
\]
where
\[
(4.8) \quad n = 2a_i x \mu + b_i (x \lambda + y \mu) + 2c_i y \lambda, \quad N = 2A_i x \mu + B_i (x \lambda + y \mu) + 2C_i y \lambda,
\]
and \(\lambda, \mu\) are integers such that
\[
(4.9) \quad \lambda x - \mu y = 1.
\]
We must show that \(T\) is well defined and that \(\text{range}(T) \subseteq C_{m, M}\). To see that \(T\) is well defined we have only to note that if \((\lambda, \mu)\) is replaced by another solution \((\lambda + ty, \mu + tx)\) of (4.10) then \(n\) and \(N\) are unchanged \((\text{mod } 2)\), and if \((x, y)\) is replaced by \((-x, -y)\) then \((\lambda, \mu)\) can be replaced by \((-\lambda, -\mu)\) and \(n\) and \(N\) remain the same.

Next we show that \(T\) maps into \(C_{m, M}\). By the transformation
\[
\begin{pmatrix} x & \mu \\ y & \lambda \end{pmatrix}
\]
the pair of forms \(((a_i, b_i, c_i), (A_j, B_j, C_j))\) becomes the pair \(((m, n, l), (M, N, L))\), where
\[
(4.11) \quad l = \frac{n^2 - d}{4m}, \quad L = \frac{N^2 - D}{4M},
\]
and so \(n^2 \equiv d \pmod{4m}\), \(N^2 \equiv D \pmod{4M}\). As \(\Delta = nN - 2mL - 2ML\), by (3.2) and Lemma 3, we have \(MN - mN = \pm k\).

Now we prove that \(T\) maps onto \(C_{m,M}\). Let \(((n \pmod{2m}), (N \pmod{2M})) \in C_{m,M}\) so that \(n^2 \equiv d \pmod{4m}\), \(N^2 \equiv D \pmod{4M}\), \(MN - mN = \pm k\). We define integers \(l, L\) as in (4.11). The forms \((m, n, l)\) and \((M, N, L)\) have discriminants \(d\) and \(D\), respectively, and, by Lemma 3 and (3.2), their codiscriminant is \(\Delta\). Hence, for a unique integer \(i (1 \leq i \leq h(d, D, \Delta))\), we have
\[
((m, n, l), (M, N, L)) \sim ((a_i, b_i, c_i), (A_i, B_i, C_i)).
\]
If
\[
\begin{pmatrix} x & \mu \\ y & \lambda \end{pmatrix},
\]
where \(\lambda x - \mu y = 1\), is a transformation mapping \(((a_i, b_i, c_i), (A_i, B_i, C_i))\) into \(((m, n, l), (M, N, L))\) then \([x, y] \in S_i\), and \(T([x, y], i)) = (n \pmod{2m}, N \pmod{2M})\). This proves that \(\text{range}(T) = C_{m,M}\).

Finally we show that \(T\) is one-to-one. Suppose that
\[
T([x, y], i) = T([x', y'], i').
\]
Then there exist integers \(n, N, n', N', t, T\) and two transformations
\[
\tau = \begin{pmatrix} x & \mu \\ y & \lambda \end{pmatrix} (x\lambda - \mu y = 1), \quad \tau' = \begin{pmatrix} x' & \mu' \\ y' & \lambda' \end{pmatrix} (x'\lambda' - y'\mu' = 1)
\]
such that
\[
(4.12) \quad n = n' + 2tm, \quad N = N' + 2TM,
\]
\[
(4.13) \quad ((a_i, b_i, c_i), (A_i, B_i, C_i)) \xrightarrow{\tau} ((m, n, l), (M, N, L)),
\]
\[
(4.14) \quad ((a_i, b_i, c_i), (A_i, B_i, C_i)) \xrightarrow{\tau'} ((m, n', l'), (M, N', L')),
\]
\[
(4.15) \quad Mn - mN = \pm k, \quad Mn' - mN' = \pm k,
\]
where \(l, L\) are defined as in (4.11), and \(l', L'\) are defined similarly. Clearly \(MN - mN = \pm(Mn' - mN')\) and we show that
\[
(4.16) \quad Mn - mN = Mn' - mN'.
\]
For otherwise \(Mn - mN = -(Mn' - mN')\) and appealing to (4.12) we obtain \(mM(T - t) = Mn - mN\). As \(\gcd(m, M) = 1\) we see that \(m|n\) and \(M|N\), and so by (4.15) we have \(mM|k\). Hence from (3.2) we have \(m|d\) and \(M|d\), contradicting \(\gcd(m, 2d) = \gcd(M, 2D) = 1\). This proves (4.16). From (4.12) and (4.16) we deduce that \(t = T\) and so
\[
\theta = \begin{pmatrix} 1 & 2t \\ 0 & 1 \end{pmatrix}
\]
maps \(((m,n,l),(M,N,L)) \rightarrow ((m',n',l'),(M',N',L'))\), proving that \(i = i'\), and that \(\tau'^{-1}\theta\tau\) is an automorphism of the pair \(((a_i,b_i,c_i),(A_i,B_i,C_i))\).

Hence by Lemma 2 we have
\[
\begin{pmatrix}
1 & 2t \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
= \mu
\begin{pmatrix}
x' \\
y'
\end{pmatrix}
= \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
x' \\
y'
\end{pmatrix},
\]
implying \([x',y'] = [x,y]\). This completes the proof that \(T\) is one-to-one.

Thus \(T\) is a bijection from \(S\) to \(C_{m,M}\) and so by (4.6) and (4.7) we have
\[
\frac{1}{2}
\Psi_{d,D,\Delta}(m,M) = \text{card}(S) = \text{card}(C_{m,M}) = \begin{cases}
2, & \text{if } k \neq 0, \\
1, & \text{if } k = 0,
\end{cases}
\]
completing the proof of Theorem (b).

5. Proof of Theorem (c)

Throughout this section we assume that \(m,M\) are positive integers satisfying (3.2) and (2.11).

First we show that if \(GCD(m,M) \neq l^2\) for any integer \(l\) then \(\Psi_{d,D,\Delta}(m,M) = 0\). For suppose \(\Psi_{d,D,\Delta}(m,M) \geq 1\). Then there exists \(i (1 \leq i \leq h(d,D,\Delta))\) and integers \(x,y\) such that
\[
m = a_i x^2 + b_i xy + c_i y^2, \tag{5.1}
\]
\[
M = A_i x^2 + B_i xy + C_i y^2.
\]
Also from (3.3) we have
\[
\pm k = (a_1 B_i - b_i A_i)x^2 + 2(a_i C_i - c_i A_i)xy + (b_i C_i - c_i B_i)y^2. \tag{5.2}
\]
Solving (5.1) and (5.2) for \(x^2, xy\) and \(y^2\), we obtain
\[
(\Delta^2 - dD)x^2 = 2(c_i D - C_i \Delta)m + 2(C_i d - c_i \Delta)M \mp 2k(b_i C_i - c_i B_i), \tag{5.3}
\]
\[
(\Delta^2 - dD)xy = (b_1 \Delta - b_i D)m + (b_i \Delta - b_i \Delta)M \pm 2k(a_i C_i - c_i A_i),
\]
\[
(\Delta^2 - dD)y^2 = 2(a_i D - A_i \Delta)m + 2(A_i d - a_i \Delta)M \pm 2k(a_i B_i - b_i A_i).
\]
As \(GCD(m,M)\) is not a square, there exists a prime \(p\) and a non-negative integer \(r\) such that \(p^{2r+1} || GCD(m,M)\). As \(m\) and \(M\) are odd we have \(p \neq 2\). Further from (3.2) we see that \(p^{2r+1} | k\) and so from (5.3) we have
\[
p^{2r+1} | (\Delta^2 - dD)x^2, \quad p^{2r+1} | (\Delta^2 - dD)y^2. \tag{5.4}
\]
By (2.11) we have \(p^r \Delta^2 - dD\) and so \(p^{r+1} | x\) and \(p^{r+1} | y\). Thus from (5.4) we have \(p^{2r+2} | m\) and \(p^{2r+2} | M\) contradicting \(p^{2r+1} || GCD(m,M)\).

Finally, if \(GCD(m,M) = l^2\), for some integer \(l\), then it is easy to check using (5.1), (5.2), and (5.3) that the mapping \((x,y) \rightarrow (x/l, y/l)\) is a bijection from the set of representations of \((m,M)\) by a set of inequivalent pairs of
forms with invariants $d, D, \Delta$ and the set of representations of $(m/l^2, M/l^2)$ by the same set of pairs of forms. Thus we have, by Theorem (b),

$$\Psi_{d,D,\Delta}(m,l) = \Psi_{d,D,\Delta}(m/l^2, M/l^2) = \begin{cases} 4, & \text{if } k/l^2 \neq 0, \\ 2, & \text{if } k/l^2 = 0, \end{cases}$$

as required. This completes the proof of Theorem (c). □

6. AN EXAMPLE

We take $d = -11, D = -11, \Delta = -19$ so that $\Delta^2 - dD = 240$. Every pair of forms with these invariants is equivalent to exactly one of the pairs

$$((1,1,3), (3,1,1)), \quad ((1,1,3), (3,5,3)), \quad ((1,1,3), (1,-3,5)), \quad ((1,1,3), (1,5,9)),$$

so $h(-11, -11, -19) = 4$.

If we take $m = 97$ and $M = 31$ (so that $GCD(m, M) = GCD(m, 2d) = GCD(M, 2D) = 1$) we have $dM^2 - 2\Delta Mm + Dm^2 = 196$, so $k = \pm 14$. Thus by Theorem (b) we must have $\Psi_{-11,-11,-19}(97, 31) = 4$. Indeed

$$97 = x^2 + xy + 3y^2, \quad 31 = x^2 - 3xy + 5y^2, \quad \text{with } (x, y) = (7, 3),$$

$$97 = x^2 + xy + 3y^2, \quad 31 = x^2 + 5xy + 9y^2, \quad \text{with } (x, y) = (10, -3).$$

Finally, we remark that the choice $m = M = 3$ shows that the condition $GCD(m, \Delta^2 - dD) = GCD(M, \Delta^2 - dD) = 1$ is necessary in Theorem (c) as

$$3 = x^2 + xy + 3y^2 = 3x^2 + xy + y^2$$

is solvable with $(x, y) = (1, -1)

REFERENCES


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