CYCLIC QUARTIC FIELDS WITH RELATIVE INTEGRAL Bases OVER THEIR QUADRATIC SUBFIELDS

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ABSTRACT. Explicit conditions are given for a cyclic quartic field to have a relative integral basis over its unique quadratic subfield.

Throughout this paper, \( K \) denotes a cyclic quartic extension of the rational number field \( Q \). By Theorem 1 of [3] we know that \( K \) can be expressed uniquely in the form

\[
K = Q\left(\sqrt{A(D + B\sqrt{D})}\right),
\]

where \( A, B, C, D \) are integers such that

- \( A \) is squarefree and odd,
- \( D = B^2 + C^2 \) is squarefree, \( B > 0, \ C > 0, \)
- \( (A, D) = 1 \).

\( K \) possesses a unique quadratic subfield \( k = Q(\sqrt{D}) \). Although \( K \) possesses an integral basis over \( Q \) (an explicit integral basis is given in [4]) it may or may not have a relative integral basis (RIB) over \( k \). In this paper we give a necessary and sufficient condition for \( K \) to have a RIB over \( k \). This is done by using the integral basis for \( K \) over \( Q \) given in [4] to determine the relative discriminant \( d(K/k) \) (see Lemma 2 below) and then appealing to the following theorem of Mann [6, Theorem 2].

**Theorem (Mann).** Let \( F \) be an algebraic number field and \( E \) a quadratic extension of \( F \). Then \( E \) has a RIB over \( F \) if and only if \( E = F(\sqrt{\Delta}) \) for some \( \Delta \in F \) with \( d(E/F) = (\Delta) \).

Our necessary and sufficient condition for \( K \) to have a RIB over \( k \) is given in terms of the fundamental unit \( \varepsilon \ (> 1) \) of \( k = Q(\sqrt{D}) \). Two cases naturally arise according as the norm \( N_{k/Q}(\varepsilon) = +1 \) or \(-1 \). First we prove

**Theorem 1.** If \( N_{k/Q}(\varepsilon) = +1 \) then \( K \) does not have a RIB over \( k \).

If \( N_{k/Q}(\varepsilon) = -1 \) we let \((U, V)\) be the solution in positive integers of \( U^2 - DV^2 = -1 \) with \( V \) least. Setting \( \varepsilon = (x + y\sqrt{D})/2 \), where \( x \) and \( y \) are positive integers.

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with \( x \equiv y \pmod{2} \), \( x^2 - Dy^2 = -4 \), we have
\[
U + V \sqrt{D} = \begin{cases} 
\varepsilon, & \text{if } x \equiv y \equiv 0 \pmod{2}, \\
\varepsilon^3, & \text{if } x \equiv y \equiv 1 \pmod{2}.
\end{cases}
\]
We note that the case \( x \equiv y \equiv 1 \pmod{2} \) can only occur when \( D \equiv 5 \pmod{8} \). It is a classical result (see for example [7, Theorem 5.9]) that if \( V > 1 \) there is a unique pair of nonnegative coprime integers \((S, T)\) such that
\[
(2) \quad V = S^2 + T^2, \quad T \equiv SU \pmod{V}.
\]
If \( V = 1 \) we take \( S = 1, \, T = 0 \) so that (2) is satisfied in this case too. A familiar argument shows that \( S \) and \( T \) satisfy the congruence
\[
(S^2 - T^2) + 2STU \equiv 0 \pmod{V^2},
\]
so that we may define nonnegative integers \( M \) and \( N \) by
\[
M = \frac{|U(S^2 - T^2) - 2ST|}{V^2}, \quad N = \frac{|(S^2 - T^2) + (2ST)U|}{V^2}.
\]
As \( M^2 + N^2 = D \), and \( D (> 1) \) is squarefree, \( M \) and \( N \) must be positive integers. Moreover we have
\[
(3) \quad (U + V \sqrt{D})(X + Y \sqrt{D})^2 = (\pm M) + \sqrt{D},
\]
where
\[
X = (T - SU)/V, \quad Y = S.
\]
We prove

**Theorem 2.** If \( N_{k/Q}(\varepsilon) = -1 \) then \( K \) has a RIB over \( k \) if and only if
\[
(B, C) = \begin{cases} 
(N, M), & \text{if } D \equiv 1 \pmod{4}, \, B \equiv 1 \pmod{2}, \\
(M, N), & \text{otherwise}.
\end{cases}
\]

It follows from Theorems 1 and 2 that if \( A \) and \( D \) are given integers such that
\[
A \text{ is squarefree and odd,}
\]
\[
D (> 1) \text{ is squarefree and representable as the sum of two squares,}
\]
\[
GCD(A, D) = 1,
\]
then either there are no pairs of positive integers \((B, C)\) with \( B^2 + C^2 = D \) for which \( K = Q(\sqrt{A(D + B\sqrt{D})}) \) has a RIB over \( k \) or there are one or two pairs according as \( D \equiv 2 \pmod{8} \) or \( D \equiv 1 \pmod{4} \).

**Example 1.** \( K = Q(\sqrt{5} + 2\sqrt{5}) \). Here \( A = 1, \, B = 2, \, C = 1, \, D = 5, \)
\[
k = Q(\sqrt{5}), \quad \varepsilon = (1 + \sqrt{5})/2, \quad N_{k/Q}(\varepsilon) = -1, \quad U = 2, \quad V = 1, \quad S = 1, \quad T = 0, \quad M = 2, \quad N = 1,
\]
and so, by Theorem 2, \( K \) has a RIB over \( k \).

**Example 2.** \( K = Q(\sqrt{10} + \sqrt{10}) \). Here \( A = 1, \, B = 1, \, C = 3, \, D = 10, \)
\[
k = Q(\sqrt{10}), \quad \varepsilon = 3 + \sqrt{10}, \quad N_{k/Q}(\varepsilon) = -1, \quad U = 3, \quad V = 1, \quad S = 1, \quad T = 0, \quad M = 3, \quad N = 1,
\]
and so, by Theorem 2, \( K \) does not have a RIB over \( k \).

In the case when \( K \) has a RIB over \( k \) we give an explicit relative integral basis for \( K/k \).
THEOREM 3. If K has a RIB over k then a RIB over K/k is given by
\[ \left\{ 1, \sqrt{A \sqrt{D}(U + V \sqrt{D})} \right\}, \quad \text{if } D \equiv 2 \pmod{8} \text{ or } D \equiv 1 \pmod{4}, \]
\[ B \equiv 0 \pmod{2}, \quad A + B \equiv 3 \pmod{4}; \]
\[ \left\{ 1, \sqrt{2A \sqrt{D}(U + V \sqrt{D})} \right\}, \quad \text{if } D \equiv 1 \pmod{4}, \quad B \equiv 1 \pmod{2}; \]
\[ \left\{ 1, \frac{1}{2}(1 + \sqrt{A \sqrt{D}(U + V \sqrt{D})}) \right\}, \quad \text{if } D \equiv 1 \pmod{4}, \]
\[ B \equiv 0 \pmod{2}, \quad A + B \equiv 1 \pmod{4}. \]

EXAMPLE 1 (CONT). Set \( \alpha = \sqrt{5 + 2\sqrt{5}}, \beta = \sqrt{5 - 2\sqrt{5}} \), so that
\[ \sqrt{5}\alpha = 2\alpha + \beta, \quad \sqrt{5}\beta = \alpha - 2\beta. \]
By Theorem 3 a RIB for \( K = Q(\sqrt{5 + 2\sqrt{5}}) \) over \( k = Q(\sqrt{5}) \) is given by \( \{1, \alpha\} \). This is easily seen directly, as every integer of \( K \) is of the form (see [4])
\[
x + y \left( \frac{1 + \sqrt{5}}{2} \right) + z \left( \frac{\alpha + \beta}{2} \right) + w \left( \frac{\alpha - \beta}{2} \right)
= \left( x + y \left( \frac{1 + \sqrt{5}}{2} \right) \right) \left( 1 + (2w - z) + (z - w) \left( \frac{1 + \sqrt{5}}{2} \right) \right) \alpha,
\]
where \( x, y, z, w \) are integers.

EXAMPLE 2 (CONT). We show directly that \( K = Q(\sqrt{10 + \sqrt{10}}) \) does not possess a RIB over \( k = Q(\sqrt{10}) \). We set \( \alpha = \sqrt{10 + \sqrt{10}}, \beta = \sqrt{10 - \sqrt{10}} \), so that
\[ \sqrt{10}\alpha = \alpha + 3\beta, \quad \sqrt{10}\beta = 3\alpha - \beta. \]
The integers of \( K \) are of the form (see [4]) \( x + y\sqrt{10} + z\alpha + w\beta \), where \( x, y, z, w \) are integers. Suppose that \( K \) has a relative integral basis over \( k \). Such a basis may be taken in the form \( \{1, \gamma\} \), where \( \gamma = t\alpha + u\beta \) with integers \( t \) and \( u \) not both zero. Thus there must be integers \( a, b, c, d, e, f, g, h \) such that
\[
\alpha = (a + b\sqrt{10})1 + (c + d\sqrt{10})(t\alpha + u\beta), \]
\[
\beta = (e + f\sqrt{10})1 + (g + h\sqrt{10})(t\alpha + u\beta),
\]
and so we have
\[
\alpha = a + b\sqrt{10} + (tc + (t + 3u)d)\alpha + (uc + (3t - u)d)\beta,
\]
\[
\beta = e + f\sqrt{10} + (tg + (t + 3u)h)\alpha + (ug + (3t - u)h)\beta.
\]
Equating coefficients of \( 1, \sqrt{10}, \alpha, \beta \), we obtain \( a = b = e = f = 0 \), and
\[
\begin{cases}
 tc + (t + 3u)d = 1, \\
 uc + (3t - u)d = 0,
\end{cases}
\quad
\begin{cases}
 tg + (t + 3u)h = 0, \\
 ug + (3t - u)h = 1.
\end{cases}
\]
Solving for \( c, d \) and \( g, h \), we obtain
\[
c = \frac{3t - u}{3t^2 - 2tu - 3u^2}, \qquad d = \frac{-u}{3t^2 - 2tu - 3u^2},
\]
\[
g = \frac{-t}{3t^2 - 2tu - 3u^2}, \qquad h = \frac{t}{3t^2 - 2tu - 3u^2}.\]
Note that \(3t^2 - 2tu - 3u^2 \neq 0\) as \(t\) and \(u\) are not both zero. As \(c, d, g, h\) are integers, we must have
\[
3t^2 - 2tu - 3u^2 \mid t, \quad 3t^2 - 2tu - 3u^2 \mid u.\]
Thus there are integers \(r\) and \(s\) such that
\[
t = (3t^2 - 2tu - 3u^2)r, \quad u = (3t^2 - 2tu - 3u^2)s,
\]
and so
\[
3t^2 - 2tu - 3u^2 = (3t^2 - 2tu - 3u^2)(3r^2 - 2rs - 3s^2),
\]
giving
\[
(3t^2 - 2tu - 3u^2)(3r^2 - 2rs - 3s^2) = 1.
\]
Hence we have
\[
3t^2 - 2tu - 3u^2 = \pm 1,
\]
and so
\[
(3t - u)^2 - 10u^2 = \pm 3,
\]
which is impossible as \(x^2 \equiv \pm 3 \pmod{5}\) is insolvable.

We now begin the proofs of Theorems 1 and 2. We first calculate the relative different \(D(K/k)\). We set
\[
\alpha = \sqrt{A(D + B\sqrt{D})}, \quad \beta = \sqrt{A(D - B\sqrt{D})}.
\]

**Lemma 1.**
\[
D(K/k) = \begin{cases} 
2(\alpha, \beta), & \text{if } B \equiv 1 \pmod{2}, \\
(\alpha + \beta, \alpha - \beta), & \text{if } B \equiv 0 \pmod{2}, A + B \equiv 3 \pmod{4}, \\
\left(\frac{\alpha + \beta}{2}, \frac{\alpha - \beta}{2}\right), & \text{if } B \equiv 0 \pmod{2}, A + B \equiv 1 \pmod{4}.
\end{cases}
\]

**Proof.** We just give the details in the case \(D \equiv 2 \pmod{8}\) (so that \(B \equiv C \equiv 1 \pmod{2}\)) as the other cases can be treated similarly. We will obtain \(D(K/k)\) from the relation
\[
D(K/k)D(k/Q) = D(K/Q).
\]
We first calculate \(D(K/Q)\). An integral basis for \(K/Q\) in this case is given by (see [4]) \(\{1, \sqrt{D}, \alpha, \beta\}\). For convenience we set \(\Omega_1 = 1, \Omega_2 = \sqrt{D}, \Omega_3 = \alpha, \Omega_4 = \beta\), and define ideals \(X_1, X_2, X_3\) of the ring \(O_K\) of integers of \(K\) by
\[
X_j = (\Omega_1 - \theta^j(\Omega_1), \quad \Omega_2 - \theta^j(\Omega_2), \quad \Omega_3 - \theta^j(\Omega_3), \quad \Omega_4 - \theta^j(\Omega_4)),
\]
where \(\text{Gal}(K/Q) = \langle \theta \rangle\), so that \(D(K/Q) = X_1X_2X_3\). As \(\theta(\alpha) = \beta, \theta(\beta) = -\alpha, \theta(\sqrt{D}) = -\sqrt{D}\), we have
\[
X_1 = X_3 = (2\sqrt{D}, \alpha - \beta, \alpha + \beta) \quad \text{and} \quad X_2 = 2(\alpha, \beta).
\]
Next, making use of
\[
\alpha^2 = AD + AB\sqrt{D}, \quad \beta^2 = AD - AB\sqrt{D}, \quad \alpha\beta = AC\sqrt{D},
\]
we obtain
\[
X_1X_3 = (4D, (\alpha - \beta)^2, (\alpha + \beta)^2, 2\sqrt{D}(\alpha - \beta), 2\sqrt{D}(\alpha + \beta), \alpha^2 - \beta^2) = 2\sqrt{D}I,
\]
where
\[ I = (2\sqrt{D}, AC + A\sqrt{D}, AC - A\sqrt{D}, \alpha - \beta, \alpha + \beta, AB). \]
Now \(2D \in I, AB \in I,\) so as \((A, D) = 1, (B, D) = 1, A \equiv B \equiv 1 \pmod{2},\) we have \((2D, AB) = (1),\) so that \(I = (1),\) and \(X_1X_3 = (2\sqrt{D}).\) Hence we have
\[ \mathcal{D}(K/Q) = (2)^2(\sqrt{D})(\alpha, \beta). \]
Next we calculate \(\mathcal{D}(k/Q).\) An integral basis for \(k\) in this case is \(\{1, \sqrt{D}\}\) and, by the definition of the different, we have
\[ \mathcal{D}(k/Q) = (1 - \theta(1), \sqrt{D} - \theta(\sqrt{D})) \]
so that
\[ \mathcal{D}(k/Q) = (2\sqrt{D}). \]
Thus, from (4), (5), (6), we obtain
\[ \mathcal{D}(K/k) = \frac{(2)^2(\sqrt{D})(\alpha, \beta)}{(2)(\sqrt{D})} = (2)(\alpha, \beta). \]
This completes the proof of Lemma 1 in this case.

Next we determine the relative discriminant \(d(K/k).\)

**Lemma 2.**
\[
d(K/k) = \begin{cases} 
(2^3A\sqrt{D}), & \text{if } D \equiv 1 \pmod{4}, B \equiv 1 \pmod{2}, \\
(2^2A\sqrt{D}), & \text{if } D \equiv 2 \pmod{8}, \text{ or } \\
(A\sqrt{D}), & \text{if } D \equiv 1 \pmod{4}, B \equiv 0 \pmod{2} A + B \equiv 3 \pmod{4}, \\
&D \equiv 0 \pmod{2} A + B \equiv 1 \pmod{4}.
\end{cases}
\]

**Proof.** We just give the details when \(D \equiv 2 \pmod{8},\) as the other cases can be treated similarly. We have (appealing to Lemma 1)
\[
d(K/k) = N_{K/k}(\mathcal{D}(K/k)) = (2)(\alpha, \beta)(2)(\beta, -\alpha) = (2)^2(\alpha, \beta)^2
= (2)^2(\alpha^2, \alpha \beta, \beta^2) = (2)^2(AD + AB\sqrt{D}, AC\sqrt{D}, AD - AB\sqrt{D})
= (2^2A\sqrt{D})(\sqrt{D} + B, \sqrt{D} - B, C).
\]
Now, as \((2B, C) = 1,\) we see that \((\sqrt{D} + B, \sqrt{D} - B, C) = (1),\) and so
\[ d(K/k) = (2^2A\sqrt{D}) \]
as required.

**Proof of Theorem 1.** Suppose \(N_{k/Q}(\epsilon) = +1\) and \(K\) has a RIB over \(k.\) Then, by Lemma 2 and Mann’s theorem, there exists \(\Delta \in \mathcal{O}_k\) such that \(K = Q(\sqrt{\Delta}),\) and \((\Delta) = (2^jA\sqrt{D}),\) where
\[
j = \begin{cases} 
3, & \text{if } D \equiv 1 \pmod{4}, B \equiv 1 \pmod{2}, \\
2, & \text{if } D \equiv 2 \pmod{8}, \text{ or } \\
& \text{if } D \equiv 1 \pmod{4}, B \equiv 0 \pmod{2}, A + B \equiv 3 \pmod{4}, \\
0, & \text{if } D \equiv 1 \pmod{4}, B \equiv 0 \pmod{2}, A + B \equiv 1 \pmod{4}.
\end{cases}
\]
Hence there is a unit \(\eta \in \mathcal{O}_k\) such that
\[ \Delta = 2^jA\sqrt{D}\eta. \]
By Dirichlet’s unit theorem we have
\[ \eta = \pm \epsilon^m, \quad \text{for some integer } m, \]
and so
\[ Q(\sqrt{A(D + B\sqrt{D})}) = Q(\sqrt{\pm 2^l A\sqrt{D}\epsilon^m}). \]
Removing squares from under the radical sign on the right-hand side as appropriate and recalling that \( Q(\sqrt{A(D + B\sqrt{D})}) \) is a cyclic field, we see that
\[ Q(\sqrt{A(D + B\sqrt{D})}) = Q(\sqrt{\pm 2^l A\sqrt{D}\epsilon}), \]
where \( l = 0, 1 \). Moreover, as \( Q(\sqrt{A(D + B\sqrt{D})}) \) and \( Q(\sqrt{\pm 2^l A\sqrt{D}\epsilon}) \) must both be totally real or both totally imaginary, we have
\[ Q(\sqrt{A(D + B\sqrt{D})}) = Q(\sqrt{2^l A\sqrt{D}\epsilon}). \]
Hence there exist \( \alpha, \beta \in k \) such that
\[ (7) \quad \sqrt{2^l A\sqrt{D}\epsilon} = \alpha + \beta \sqrt{A(D + B\sqrt{D})}. \]
From (7) we see that
\[ \sqrt{2^l A\sqrt{D}\epsilon} \sqrt{A(D + B\sqrt{D})} = \frac{1}{2\beta} \left( 2^l A\sqrt{D}\epsilon + \beta^2 A(D + B\sqrt{D}) - \alpha^2 \right) \in Q(\sqrt{D}). \]
Hence there exist rational numbers \( e \) and \( f \) such that
\[ (8) \quad \sqrt{2^l A\sqrt{D}\epsilon} \sqrt{A(D + B\sqrt{D})} = e + f \sqrt{D}. \]
Squaring (8) and taking norms, we obtain
\[ 2^{2l} A^2 (-D) A^2 D C^2 = \left( e^2 - D f^2 \right)^2, \]
which is impossible. This completes the proof of Theorem 1.

**Lemma 3.** If \( N_{k/Q}(\epsilon) = -1 \) then \( K \) has a RIB over \( k \) if and only if
\[ K = \begin{cases} 
Q(\sqrt{2A\sqrt{D}(U + V\sqrt{D})}), & \text{if } D \equiv 1 \pmod{4}, \ B \equiv 1 \pmod{2}, \\
Q(\sqrt{A\sqrt{D}(U + V\sqrt{D})}), & \text{otherwise}. 
\end{cases} \]

**Proof.** We just treat the case \( D \equiv 1 \pmod{4}, \ B \equiv 1 \pmod{2} \), as the other cases can be treated similarly. By Lemma 2 and Mann’s theorem, \( K \) has a RIB over \( k \) if and only if
\[ (9) \quad K = Q(\sqrt{23 A\sqrt{D}\lambda}), \]
for some positive unit \( \lambda \) in \( O_k \). The unit \( \lambda \) must be positive for if \( \lambda \) were negative \( Q(\sqrt{A(D + B\sqrt{D})}) \) and \( Q(\sqrt{23 A\sqrt{D}\lambda}) \) could not both be totally real or both totally imaginary. By Dirichlet’s unit theorem, we have \( \lambda = \epsilon^m \) for some integer \( m \). Recalling that \( U + V\sqrt{D} = \epsilon \) or \( \epsilon^3 \) and removing squares from under the radical sign in (9) we see that \( K \) has a RIB over \( k \) if and only if
\[ K = Q(\sqrt{2A\sqrt{D}(U + V\sqrt{D})^j}), \]
where \( j = 0 \) or \( 1 \). As \( K \) is cyclic we must have \( j = 1 \). This completes the proof of Lemma 3 in this case.
LEMMA 4. If $N_{k/Q}(\epsilon) = -1$, then we have
\[ Q(\sqrt{2A\sqrt{D}(U + V\sqrt{D})}) = Q(\sqrt{A(D + N\sqrt{D})}), \]
\[ Q(\sqrt{A\sqrt{D}(U + V\sqrt{D})}) = Q(\sqrt{A(D + M\sqrt{D})}). \]

PROOF. This is clear from (3) and the fact
\[ Q(\sqrt{2A\sqrt{D}(M + \sqrt{D})}) = Q(\sqrt{A\sqrt{D}(N + \sqrt{D})}). \]

PROOF OF THEOREM 2. Theorem 2 follows immediately from Lemmas 3 and 4 as the representation of $K$ in the form (1) is unique.

PROOF OF THEOREM 3. In each case it is a simple matter to check that the given set of elements has discriminant equal to $d(K/k)$ (the value of which is given in Lemma 2). Appealing to Theorem 2 and Lemma 4, it is easy to check that in each case the elements lie in $K$. The only element which is not obviously an algebraic integer is
\[ \gamma = \frac{1}{2}(1 + \sqrt{A\sqrt{D}(U + V\sqrt{D})}). \]

Since $\gamma$ satisfies
\[ \gamma^2 - \gamma + \frac{1}{4}(1 - AVD - AU\sqrt{D}) = 0 \]
it suffices to show that $\frac{1}{4}(1 - AVD - AU\sqrt{D})$ is an integer of $k$. Since $D \equiv 1 \pmod{4}$ (in this case) and $U^2 - V^2D = -1$, we have $U \equiv 0 \pmod{2}$ and $V \equiv 1 \pmod{2}$. Moreover we have $V \equiv 1 \pmod{4}$ as $U^2 \equiv -1 \pmod{V}$. Hence $1 - AVD$ and $AU$ are both even, and so, it suffices to show that
\[ 1 - AVD \equiv -AU \pmod{4}, \]
or equivalently
\[ A(VD - U) \equiv 1 \pmod{4}. \]

We consider two cases according as $U \equiv 0 \pmod{4}$ or $U \equiv 2 \pmod{4}$. If $U \equiv 0 \pmod{4}$ then, from $U^2 - V^2D = -1$, we deduce that $V^2D \equiv 1 \pmod{8}$, so that $D \equiv 1 \pmod{8}$, and thus $B \equiv 0 \pmod{4}$. Hence, as $A + B \equiv 1 \pmod{4}$ (in this case), we obtain $A \equiv 1 \pmod{4}$, giving $A(VD - U) \equiv 1 \pmod{4}$. If $U \equiv 2 \pmod{4}$ then as above we conclude $D \equiv 5 \pmod{8}, B \equiv 2 \pmod{4}, A \equiv 3 \pmod{4}$ and $A(VD - U) \equiv 1 \pmod{4}$. This completes the proof of Theorem 3.

We conclude by remarking that Xianke [8] has given a less explicit form of Theorems 1, 2, 3. Relative integral bases for bicyclic quartic fields over their quadratic subfields are considered in [1, 2 and 5].

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