THE CLASS NUMBER TWO PROBLEM FOR CERTAIN QUARTIC FIELDS

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1. Introduction. Let K denote an algebraic number field of finite degree over the rational field Q. The ring of integers of K is denoted by O_K. If A and B are nonzero ideals of O_K, we say that A is equivalent to B, written $A \sim B$, if there exist nonzero elements α and β of O_K such that $(\alpha)A = (\beta)B$. It is easy to check that $\sim$ is an equivalence relation and it is a classical result that the number of equivalence classes is finite. The number of equivalence classes is called the classnumber of K and is denoted by $h(K)$.

It is a result going back to Dedekind that $h(K) = 1$ if and only if O_K is a unique factorization domain. More recently Carlitz [5] has shown that $h(K) = 2$ if and only if O_K is not a unique factorization domain but every factorization of a nonzero, nonunit integer of K contains the same number of primes. It is thus of interest to determine those algebraic number fields K having $h(K) = 1$ or $h(K) = 2$. However this is an extremely difficult problem. Even if K is restricted to a certain class of fields, such as quadratic fields, the problem is still difficult.

The first results of this type were obtained by Stark [11] in 1967 who showed that there are exactly nine imaginary quadratic fields $K = Q(\sqrt{d})$ ($d < 0, d$ squarefree) with classnumber 1, namely those for which $d = -1, -2, -3, -7, -11, -19, -43, -67$ or -163. The determination of all imaginary quadratic fields $K = Q(\sqrt{d})$ ($d < 0, d$ squarefree) with $h(K) = 2$ was carried out by Baker [1] and Stark [12] in 1971. They proved that

$$\begin{aligned} h(K) = 2 \Leftrightarrow d = &-5, -6, -10, -15, -22, -35, -37 \\ &-51, -52, -58, -91, -115, -123, \\ &-187, -235, -267, -403, -427. \end{aligned}$$

More recently Mestre [9] has shown that if $-d$ is prime then

$$h(Q(\sqrt{d})) > \frac{1}{55} \log \mid d \mid,$$

with a similar inequality when $-d$ is composite. These inequalities allow in principle the determination of all imaginary quadratic fields $K = Q(\sqrt{d})$ ($d < 0, d$ squarefree) having $h(K) \leq 100$. There are 16 imaginary quadratic fields with $h(K) = 3$ and 54 fields with $h(K) = 4$. These results for imaginary quadratic fields contrast sharply with the case when $k = Q(\sqrt{d})$ is a real quadratic field. It was conjectured by Gauss that there are infinitely many real quadratic fields K for which $h(K) = 1$ but it is still not known whether this is true or false.

In the case of imaginary bicyclic quartic fields $K = Q(\sqrt{d_1}, \sqrt{d_2})$, Brown and Parry [3] showed in 1974 that $h(K) = 1$ if and only if K belongs to a list of 47 fields. In 1977 Buell, Williams and Williams [4] showed that $h(K) = 2$ if and only if K belongs to a list of 160 fields, provided the known list of imaginary quadratic fields with classnumber 4 is complete. Since this list is now known to be complete from the work of Mestre mentioned above, the list of 160 imaginary bicyclic quartic fields of classnumber 2 is also complete.

In the case of imaginary cyclic quartic fields K, Uchida [13] showed in 1972 that if the conductor f of the field satisfies $f \geq 50,000$ then $h(K) > 1$. Later, in 1980, Setzer [10] examined the imaginary cyclic quartic fields K with $f < 50,000$ and determined all those wtih $h(K) = 1$. He found that

$$h(K) = 1 \Leftrightarrow f = 5, 13, 16, 29, 37, 53, 61.$$

Turning next to cyclotomic fields, Masley and Montgomery [8] in 1976 determined all cyclotomic fields $K = Q(e^{2\pi i/n})$ ($n \not\equiv 2 (\bmod 4)$) for which $h(K) = 1$. They proved that

$$\begin{aligned} h(K) = 1 \Leftrightarrow n = &3, 4, 5, 7, 8, 9, 11, 12, 13, 15, 16, 17, \\ &19, 20, 21, 24, 25, 27, 28, 32, 33, 35, \\ &36, 40, 44, 45, 48, 60, 84. \end{aligned}$$

Also in 1976 Masley [7] determined the cyclotomic fields K for which $2 \leq h(K) \leq 10$.

There are also results for other types of fields. I just mention that Uchida [13] has determined all those imaginary octic fields $Q(\sqrt{d_1}, \sqrt{d_2}, \sqrt{d_3})$ with classnumber 1. He showed that there are just 17 such fields.

The determination of all imaginary cyclic quartic fields of classnumber 2 does not appear to have been dealt with in the literature. In this talk I will describe briefly the solution to the classnumber 2 problem for these fields.

2. Cyclic quartic extensions of Q. It is shown in [6] that every cyclic quartic extension K of Q can be written in the form

$$K = Q(\sqrt{A(D+B\sqrt{D})}), \tag{2.1}$$

where

$$\begin{cases} A \text{ is squarefree and odd,} \\ D = B^2 + C^2 \text{ is squarefree}, B > 0, C > 0, \\ (A, D) = 1. \end{cases} \tag{2.2}$$

Moreover any field of the form (2.1) satisfying (2.2) is a cyclic quartic extension of Q. Further, the representation (2.1), (2.2) is unique in the sense that if $K = Q(\sqrt{A_1(D_1 + B_1\sqrt{D_1})})$ is another representation of K satisfying (2.2) then $A = A_1, B = B_1, C = C_1, D = D_1$.

In [6] the discriminant $d(K)$ of the field K is determined in terms of A, B, C, D. It is shown that

$$d(K) = 2^e A^2 D^3, \tag{2.3}$$

where

$$e = \begin{cases} 8, & \text{if } D \equiv 2 (\text{mod } 8)\ , \\ 6, & \text{if } D \equiv 1(\text{mod } 4), B \equiv 1(\text{mod } 2)\ , \\ 4, & \text{if } D \equiv 1(\text{mod } 4), B \equiv 0(\text{mod } 2), A + B \equiv 3(\text{mod } 4)\ , \\ 0 & \text{if } D \equiv 1(\text{mod } 4), B \equiv 0(\text{mod } 2), A + B \equiv 1(\text{mod } 4)\ . \end{cases} \tag{2.4}$$

By the discriminant-conductor formula we have

$$d(K) = mf^2, \tag{2.5}$$

where m is the conductor of $k = Q(\sqrt{D})$ the unique (real) quadratic subfield of K. As

$$m = \begin{cases} D, & \text{if } D \equiv 1(\text{mod } 4), \\ 4D, & \text{if } D \equiv 2(\text{mod } 8), \end{cases} \tag{2.6}$$

we have

$$f = 2^{\ell} |A| D, \tag{2.7}$$

where

$$\ell = \begin{cases} 3, & \text{if } D \equiv 2(\text{mod } 8) \text{ or } D \equiv 1(\text{mod } 4), B \equiv 1(\text{mod } 2), \\ 2, & \text{if } D \equiv 1(\text{mod } 4), B \equiv 0(\text{mod } 2), A + B \equiv 3(\text{mod } 4), \\ 0, & \text{if } D \equiv 1(\text{mod } 4), B \equiv 0(\text{mod } 2), A + B \equiv 1(\text{mod } 4). \end{cases} \tag{2.8}$$

3. Formulae for $h(K)$. Let G denote the multiplicative group of residues coprime with f so that G is isomorphic in a natural way to $Gal(Q(e^{2\pi i/f})/Q)$. We let H denote the subgroup of G, which is isomorphic to $Gal(Q(e^{2\pi i/f})/K)$. By galois theory we know that G/H is a cyclic group of order 4, say

$$(3.1) \qquad G/H = <\alpha H>$$

In what we do the particular choice of α will not be important. We define a character χ on G by

$$(3.2) \qquad \chi(\alpha) = i, \ \chi(h) = 1 \ \forall \ h \in H.$$

It is easy to show that all the characters on G, which are trivial on H, are given by

$$(3.3) \qquad \chi_0, \ \chi, \ \chi^2, \ \chi^3,$$

where $\chi^4 = \chi_0$ is the trivial character on G. The characters χ and $\chi^3 = \bar{\chi}$ are both odd primitive characters of conductor f. The character χ^2 however may not be primitive. The primitive character $(\chi^2)'$ induced by χ^2 is

$$(3.4) \qquad (\chi^2)'(n) = \left(\frac{m}{n}\right), n > 0, (n,m) = 1,$$

where m is the conductor of $k = Q(\sqrt{D})$.

For s a complex variable, we set

$$(3.5) \qquad L_1(s) \ = \ L(s,\chi) \ L(s,\chi^3)$$

and

$$(3.6) \qquad L_2(s) = L(s,\chi^2).$$

It follows from [6] that

$$\frac{h(K)}{h(k)} = \frac{f w(K) L_1(1)}{4\pi^2},$$

where $w(K)$ denotes the number of roots of unity in K, that is,

$$(3.7) \qquad w(K) = \begin{cases} 2, & \text{if } f > 5, \\ 10, & \text{if } f = 5. \end{cases}$$

Since $h(K) = 1$ when $f = 5$, we may assume that $f > 5$. As k is the maximal real subfield of K, the classnumber $h(k)$ divides the classnumber $h(K)$, and the integer $h(K)/h(k)$ is called the relative classnumber of K (over k) and is denoted by $h^*(K)$. Thus we have

$$(3.8) \qquad h^*(K) = \frac{f L_1(1)}{2\pi^2}, \ f > 5.$$

From the work of Berndt [2], we know that

(3.9)
$$L(1,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n} = \frac{\pi \sum_{0<n<f/2} \bar{\chi}(n)}{iG(\bar{\chi})\,(\chi(2)-2)},$$

where the Gauss sum $G(\chi)$ is defined by

(3.10)
$$G(\chi) = \sum_{j=1}^{f} \chi(j)\, e^{2\pi i j/f}.$$

Since

(3.11)
$$G(\chi)\, G(\bar{\chi}) = -f,$$

we obtain

(3.12)
$$L_1(1) = \frac{\pi^2}{f(\chi(2)-2)(\bar{\chi}(2)-2)} \left| \sum_{0<n<f/2} \chi(n) \right|^2$$

and so

(3.13)
$$h^*(K) = \rho \left| \sum_{0<n<f/2} \chi(n) \right|^2, \; f > 5,$$

where

(3.14)
$$\rho = \begin{cases} \frac{1}{8}, & f \text{ even}, \\ \frac{1}{2}, & f \text{ odd}, \chi(2) = 1, \\ \frac{1}{18}, & f \text{ odd}, \chi(2) = -1, \\ \frac{1}{10}, & f \text{ odd}, \chi(2) = \pm i. \end{cases}$$

Defining, for $j = 0, 1, 2, 3$,

(3.15)
$$C_j = \sum_{\substack{0<n<f/2 \\ \chi(n)=i^j}} 1 = \sum_{\substack{0<n<f/2 \\ n\in\alpha^j H}} 1,$$

we obtain

(3.16)
$$h^*(K) = \rho\{(C_0 - C_2)^2 + (C_1 - C_3)^2\}.$$

4. Lower bound for $h^*(K)$. By extending the ideas used in [13], and the formula (3.8), it can be shown that

(4.1)
$$h^*(K) > 2 \text{ for } f \geq 416{,}000.$$

Thus in order to determine all imaginary cyclic quartic fields with $h^*(K) = 2$ it suffices to consider only those having $f < 416,000$.

5. Necessary and sufficient condition for $h^*(K) \equiv 2 \pmod 4$. In searching the imaginary cyclic quartic fields K of conductor $f < 416,000$ for those fields with $h^*(K) = 2$, it suffices to calculate $h^*(K)$ only for those fields K having $h^*(K) \equiv 2 (\text{mod } 4)$. It is shown in [6] that

$$(5.1) \quad \begin{aligned} & h^*(K) \equiv 2 (\text{mod } 4) \\ & \quad \Leftrightarrow f = 16p, \text{ where } p \equiv 3 \text{ or } 5 \pmod 8, \\ & \quad \text{or } f = 8p, \text{ where } p \equiv 5 (\text{mod } 8), \\ & \quad \text{or } f = pq, \text{ where } (p/q) = -1. \end{aligned}$$

Here p and q denote distinct odd primes. This considerably reduces the number of fields K for which $h^*(K)$ must be calculated.

6. Calculation of $h^*(K)$. Using the formula for $h^*(K)$ given in (3.16) and the results of §2, $h^*(K)$ was calculated by the method described in [6] for all fields K with $f < 416,000$ and f of the form (5.1). It was found that

$$\begin{aligned} h^*(K) = 2 \Leftrightarrow K = & Q(\sqrt{-(5+\sqrt{5})}) && (f = 40) \\ & Q(\sqrt{-3(2+\sqrt{2})}) && (f = 48) \\ & Q(\sqrt{-5(13+2\sqrt{13})}) && (f = 65) \\ & Q(\sqrt{-13(5+2\sqrt{5})}) && (f = 65) \\ & Q(\sqrt{-5(2+\sqrt{2})}) && (f = 80) \\ & Q(\sqrt{-(10+3\sqrt{10})}) && (f = 80) \\ & Q(\sqrt{-17(5+2\sqrt{5})}) && (f = 85) \\ & Q(\sqrt{-(85+6\sqrt{85})}) && (f = 85) \\ & Q(\sqrt{-(13+3\sqrt{13})}) && (f = 104) \\ & Q(\sqrt{-7(17+4\sqrt{17})}) && (f = 119) \end{aligned}$$

7. Solution of classnumber 2 problem. We have

$$h(K) = 2 \Leftrightarrow h^*(K) = 2, h(k) = 1$$

or

$$h^*(K) = 1, h(k) = 2.$$

However from [10] we know that

$$h^*(K) = 1, h(k) = 2$$

cannot occur so that

$$h(K) = 2 \Leftrightarrow h^*(K) = 2, h(k) = 1.$$

Thus $h(K) = 2$ occurs only for those fields K in the list of §6 for which $h(k) = 1$. Since

$$h\left(Q(\sqrt{2})\right) = h\left(Q(\sqrt{5})\right) = h\left(Q(\sqrt{13})\right) = h\left(Q(\sqrt{17})\right) = 1$$

and

$$h\left(Q(\sqrt{10})\right) = h\left(Q(\sqrt{85})\right) = 2,$$

we have proved the following theorem.

THEOREM. Let K be an imaginary cyclic quartic field. Then $h(K) = 2$ if and only if

$$K = Q\left(\sqrt{-3(2+\sqrt{2})}\right), Q\left(\sqrt{-5(2+\sqrt{2})}\right), Q\left(\sqrt{-(5+\sqrt{5})}\right),$$
$$Q\left(\sqrt{-13(5+2\sqrt{5})}\right), Q\left(\sqrt{-17(5+2\sqrt{5})}\right), Q\left(\sqrt{-(13+3\sqrt{13})}\right),$$
$$Q\left(\sqrt{-5(13+2\sqrt{13})}\right), \text{ or } Q\left(\sqrt{-7(17+4\sqrt{17})}\right).$$

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