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An Observation on Binary Quadratic Forms of Discriminant $-32q$

By PHILIP A. LEONARD and KENNETH S. WILLIAMS*)

Abstract: It is shown that a result of the authors yields an improvement on a theorem of PIERRE BARRUCAND and HARVEY COHN.

Throughout, we let $q \equiv 1 \pmod 8$ be a fixed prime for which the class number of h of $Q(\sqrt{-2q})$ satisfies $h \equiv 4 \pmod 8$; thus, the 2-Sylow subgroup of the ideal class group of this field is cyclic of order 4. If $p \equiv 1 \pmod 8$ is a prime such that the Legendre symbol (p/q) has the value $+1$, then for an odd positive integer z dividing h we have either

$$p^z = u_0^2 + 8qv_0^2, \qquad (u_0, v_0) = 1, \tag{1}$$

or

$$p^z = 8u_1^2 + qv_1^2, \qquad (u_1, v_1) = 1. \tag{2}$$

Using results of P. KAPLAN [4], the authors have shown [5] how the parities of v_0 and u_1 in (1) and (2) are determined. We shall restate our result and show how it strengthens a theorem ([2], Theorem 5.1) of P. BARRUCAND and H. COHN.

Suppose $2q = A^2 + B^2$, and consider the Legendre symbol $((A + Bi)/p)$, where i is interpreted rationally as a solution of the congruence $x^2 \equiv -1 \pmod p$. Let $\lambda(p)$ be 0 or 1 according as the value of this symbol is $+1$ or -1.

Theorem 1 (see [5]). If (1) holds, then $v_0 \equiv \lambda(p) \pmod 2$; if (2) holds, then $u_1 \equiv \lambda(p) + (q-1)/8 + 1 \pmod 2$.

Let ε_q denote the fundamental unit of $Q(\sqrt{q})$. The 4-class field of $Q(\sqrt{-q})$ is $Q(\sqrt{-q}, \sqrt{q}, \sqrt{\varepsilon_q})$ (see for example [2]). Thus, for a prime $p = r^2 + 8s^2$ such that $(p/q) = +1$, we have $(\varepsilon_q/p) = +1$ if, and only if, the ideal class P of a prime ideal divisor of (p) in $Q(\sqrt{-q})$ is a fourth power in the ideal class group of this imaginary quadratic field. (We write $P \in H^4$ for this last condition.) On the other hand, a residuacity result for ε_q due to SCHOLZ [6] (see also [8]) and the rational biquadratic law of BURDE [3] (see also [7]) give

$$(\varepsilon_q/p) = (p/q)_4\,(q/p)_4 = ((a + 2bi)/p), \tag{3}$$

where $q = a^2 + 4b^2$. Now

$$((A + Bi)/p) = ((1 + i)/p)\,((a + 2bi)/p),$$

and

$$((1 + i)/p) = (-1)^s \qquad \text{(see [1])},$$

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so we have

$$\lambda(p) \equiv s (\text{mod } 2) \Leftrightarrow P \in H^4. \tag{4}$$

Combining (4) with Theorem 1, we obtain the following result (using the notation introduced above).

Theorem 2. If (1) holds, then $s \equiv v_0 (\text{mod } 2)$ if, and only if, $P \in H^4$. If (2) holds, then $s \equiv u_1 + (q-1)/8 + 1 (\text{mod } 2)$ if, and only if, $P \in H^4$.

Theorem 2 is to be compared to ([2], Theorem 5.1), a result identical except in (2), where Barrucand and Cohn have the congruence $s \equiv u_1 + (q-1)/8 + L''$ (mod 2), with L'' a constant, for a given q, equal to 0 or 1. Thus our assertion is that $L'' = 1$ for all primes q under consideration. (We note that example (5.19c) of [2] indicating that $L'' = 0$ for $q = 97$ is in error as $881 = 28^2 + 97.1^2$.)

The constant L'' of [2] comes into the diophantine problem we have considered by way of certain ideal congruences ([2], Theorem 4.7) describing the primes of $Q(\sqrt{-2q})$ which split in $Q(\sqrt{-2q}, \sqrt{q}, \sqrt{\pi})$, where $q = r_0^2 + 8s_0^2$ gives rise to $\pi = r_0 + 2s_0\sqrt{-2}$, normalized by requiring $r_0 + 2s_0 \equiv 1$ (mod 4). The derivation of these ideal congruences is not carried out explicitly in [2]. Our approach has been to treat the diophantine problem directly and thereby avoid the slight imprecision in the earlier calculation of L''.

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Anschriften der Autoren: Philip A. Leonard, Arizona State University, Tempe, Arizona 85287, U.S.A.; Kenneth S. Williams, Carleton University, Ottawa, Ontario, K1S 5B6 Canada.