Let \( m \) be a positive squarefree integer. We denote the class number of \( \mathbb{Q}(\sqrt{-m}) \) by \( h(-m) \) and the fundamental unit of \( \mathbb{Q}(\sqrt{-m}) \) by \( \varepsilon_m \). We consider only those \( m \) for which the norm of \( \varepsilon_m \) (written \( N(\varepsilon_m) \)) is \(-1\), so that the only possible primes dividing \( m \) are the prime 2 or primes congruent to 1 modulo 4. Now, if \( p \) is an odd prime such that \( (mlp) = +1 \), we can interpret \( \varepsilon_m \) as an integer modulo \( p \), and ask for the value of the Legendre symbol \( (\varepsilon_m/p) \). Because of the ambiguity in the choice of \( \sqrt{m} \) taken modulo \( p \), we must ensure that \( (\varepsilon_m/p) \) is well-defined. Since

\[
\left( \frac{-1}{p} \right) = \left( \frac{N(\varepsilon_m)}{p} \right) = \left( \frac{\varepsilon_m \varepsilon_m'}{p} \right) = \left( \frac{\varepsilon_m}{p} \right) \left( \frac{\varepsilon_m'}{p} \right),
\]

where the prime (') indicates conjugation (\( \sqrt{m} \to -\sqrt{m} \)), this will be the case if \( (-1/p) = +1 \), that is, if \( p \equiv 1 \mod 4 \). Thus it is assumed throughout that

\[
\left( \frac{-1}{p} \right) = \left( \frac{m}{p} \right) = +1.
\]

Suppose \( m \) has the prime decomposition \( m = q_1 \ldots q_s \), and let \( a \) denote the number of ambiguous classes of forms of discriminant \(-4m\) in the principal genus. Then, from genus theory, we know that

\[ b = \begin{cases} 2^a, & \text{if } m \text{ odd,} \\ 2^{a-1} a, & \text{if } m \text{ even,} \end{cases} \]

is an integer dividing \( h(-m) \), and we define a positive integer \( l \) by

\[ l = h(-m)/b. \]

We restrict our attention to primes (congruent to 1 modulo 4) represented by forms in genera containing ambiguous classes, so that \( p^l \) is represented by an ambiguous form. For such primes \( p \), when \( m \) is a prime or twice a prime, the evaluation of \( (\varepsilon_m/p) \) is known, except in one case. In these cases, the generic characters are given by (for \( k > 0 \))
\[ \chi_1(k) = \left( -\frac{2}{k} \right) \quad m = 2, \]
\[ \chi_1(k) = \left( -\frac{1}{k} \right), \quad \chi_2(k) = \left( \frac{k}{q} \right), \quad m = q \text{ (prime)} \equiv 1 \pmod{4}, \]
\[ \chi_1(k) = \left( -\frac{2}{k} \right), \quad \chi_2(k) = \left( \frac{k}{q} \right), \quad m = 2q, \quad q \text{ (prime)} \equiv 1 \pmod{4}, \]

and the ambiguous forms of discriminant \(-4m\) are given by

\[ \begin{align*}
I &= (1, 0, 2), \quad m = 2, \\
I &= (1, 0, q), \quad A = (2, 2, \frac{1}{2}(q + 1)), \quad m = q, \\
I &= (1, 0, 2q), \quad A = (2, 0, q), \quad m = 2q,
\end{align*} \]

where \((r, s, t)\) denotes the form \(rx^2 + sxy + ty^2\).

We remark that \(N(\varepsilon_m) = -1\) when \(m = 2\); when \(m = q \text{ (prime)} \equiv 1 \pmod{4}\) (Dirichlet [6: p. 225]); and when \(m = 2q, \quad q \text{ (prime)} \equiv 5 \pmod{8}\) (Dirichlet [6: p. 226]). \(m = 2q, \quad q \text{ (prime)} \equiv 1 \pmod{8}\) is the only case which requires the assumption that the norm of the fundamental unit be \(-1\). In this case, the assumption \(h = h(-2q) \equiv 4 \pmod{8}\) has also to be made, as Lehmer’s results [11: Theorems 2 and 3] require that \(h/4\) be odd. What happens when \(h \equiv 0 \pmod{8}\) remains open.

Both possibilities occur as \(N(\varepsilon_{2q}) = N(\varepsilon_{2q+1}) = -1, \quad h(-82) = 4, \quad h(-226) = 8\). Writing \(h\) for \(h(-m)\) the results in the known cases can be summarized as follows:

<table>
<thead>
<tr>
<th>(m)</th>
<th>Assumptions</th>
<th>Evaluation of (\left( \varepsilon_m, \frac{p}{\varepsilon_m} \right))</th>
<th>References</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 (q \equiv 1 \pmod{8})</td>
<td>(\left( -\frac{1}{p} \right) = \left( \frac{2}{p} \right) = 1)</td>
<td>((-1)^{y/2}, \text{ if } p = x^2 + 2y^2)</td>
<td>[1]</td>
</tr>
<tr>
<td>(2q \quad q \equiv 1 \pmod{8})</td>
<td>(N(\varepsilon_{2q}) = -1, \quad h \equiv 4 \pmod{8})</td>
<td>((-1)^{x/2}, \text{ if } p^h/4 = x^2 + qy^2)</td>
<td>[11]</td>
</tr>
<tr>
<td>(2q \quad q \equiv 5 \pmod{8})</td>
<td>(\left( -\frac{1}{p} \right) = \left( \frac{2q}{p} \right) = 1)</td>
<td>((-1)^{y/2}, \text{ if } p^h/2 = x^2 + 2qy^2)</td>
<td>[11] [13]</td>
</tr>
</tbody>
</table>
It is the purpose of this paper to discuss the remaining cases when \( m \) has exactly two prime factors, that is, \( m = q_1 q_2 \), where \( q_1 \) and \( q_2 \) are distinct primes congruent to 1 (mod 4).

In the unique factorization domain \( \mathbb{Z}[i] \) of Gaussian integers, we have \( q_1 = \pi_1 \bar{\pi}_1 \), \( q_2 = \pi_2 \bar{\pi}_2 \), where \( \pi_1 \) and \( \pi_2 \) are primes, which we can take to be primary, that is, to satisfy \( \pi_1 \equiv \pi_2 \equiv 1 \pmod{(1 + i)^3} \). Now either \( \varepsilon_{q_1 q_2} \) or \( \varepsilon_{q_1 q_2} \) is of the form \( T + U \sqrt{q_1 q_2} \), where \( T \) and \( U \) are positive integers with \( T \) even and \( U \) odd. Since \( N(T + U \sqrt{q_1 q_2}) = -1 \), we have, for \( j = 1, 2 \), \( \pi_j(T + i)(T - i) \), that is, \( \pi_j \mid T \pm i \), as \( \pi_j \) is prime. Replacing \( \pi_j \) by its complex conjugate \( \bar{\pi}_j \), if necessary, we can assume

\[
\pi_j \mid T + i \quad (j = 1, 2).
\]

Writing \([ \frac{1}{\pi_1} ]_2 \) (resp. \([ \frac{1}{\pi_2} ]_4 \)) for the quadratic (resp. biquadratic) residue symbol (mod \( \pi_j \)), and \( ( \frac{1}{p} )_4 \) for the rational biquadratic symbol (mod \( p \)) \((p \) an odd prime), we have

**THEOREM 1.** If \( p, q_1, q_2 \) are distinct primes congruent to 1 (mod 4), such that \( (q_1 q_2/p) = +1 \), then

\[
\left( \frac{q_1 q_2}{p} \right) = \left[ \frac{p}{\pi_1} \right]_4 \left[ \frac{p}{\pi_2} \right]_4 \left( \frac{q_1 q_2}{p} \right)_4,
\]

where \( \pi_1, \pi_2 \) are defined as above. (Compare Furuta [7: Theorem 3])

**PROOF.** As \( T \) is even, \( (T + i)/\pi_1 \pi_2 \), and \( (T - i)/\pi_1 \pi_2 \) are coprime Gaussian integers. Since their product is \( U^2 \), by the unique factorization property, \( (T + i)/\pi_1 \pi_2 \) must be an associate of a square, say,

\[
T + i = u \pi_1 \pi_2 \alpha^2,
\]

where \( u \) is a unit of \( \mathbb{Z}[i] \), that is, \( u = \pm 1, \pm i \). Reducing this equation modulo 2, we obtain \( u \equiv i \pmod{2} \), so that \( u = \pm i \). Replacing \( \alpha \) by \( i \alpha \), if necessary, we have

\[
(1)
T + i = i \pi_1 \pi_2 \alpha^2.
\]

As \( U > 0, \alpha \bar{\alpha} > 0 \), this gives \( U = \alpha \bar{\alpha} \). Hence, from

\[
2(T + i)(T + U \sqrt{q_1 q_2}) = (T + i + U \sqrt{q_1 q_2})^2,
\]

we have

\[
(2)
(1 + i)^2 \pi_1 \pi_2 (T + U \sqrt{q_1 q_2}) = (i \pi_1 \pi_2 \alpha + \bar{\alpha} \sqrt{q_1 q_2})^2.
\]

Let \( \pi \) be a primary prime factor of \( p \) in \( \mathbb{Z}[i] \), so that \( p = \pi \bar{\pi}, \pi \equiv \bar{\pi} \equiv 1 \pmod{(1 + i)^3} \). Interpreting \( \sqrt{q_1 q_2} \) as an integer modulo \( p \), we have from

(2)
\[
\left( \frac{\varepsilon_{q_1 q_2}}{p} \right) = \left( \frac{T + \sqrt{q_1 q_2}}{p} \right) = \left[ \frac{T + \sqrt{q_1 q_2}}{\pi} \right]_2
\]

\[
= \left[ \frac{\pi_1 \pi_2}{\pi} \right]_2 = \left[ \frac{\pi_1}{\pi} \right]_2 \left[ \frac{\pi_2}{\pi} \right]_2
\]

\[
= \left[ \frac{\pi}{\pi_1} \right]_4 \left[ \frac{\pi}{\pi_2} \right]_4 \text{ (by the law of quadratic reciprocity in } \mathbb{Z}[i] \text{)}
\]

\[
= \left[ \frac{\pi}{\pi_1} \right]_4 \left[ \frac{\pi}{\pi_2} \right]_4 \cdot \left[ \frac{\pi}{\pi_1} \right]_4 \left[ \frac{\pi}{\pi_2} \right]_4 \cdot \left[ \frac{\pi}{\pi_1} \right]_4 \left[ \frac{\pi}{\pi_2} \right]_4 \cdot \left[ \frac{\pi}{\pi_1} \right]_4 \left[ \frac{\pi}{\pi_2} \right]_4
\]

\[
= \left[ \frac{p}{\pi_1} \right]_4 \left[ \frac{p}{\pi_2} \right]_4 \left( \frac{q_1 q_2}{\pi} \right)_4 \text{ (by the law of biquadratic reciprocity in } \mathbb{Z}[i] \text{)}
\]

\[
= \left[ \frac{p}{\pi_1} \right]_4 \left[ \frac{p}{\pi_2} \right]_4 \left( \frac{q_1 q_2}{p} \right)_4.
\]

**Corollary 1.** If \( p, q_1, q_2 \) are distinct primes congruent to 1 modulo 4, such that \((q_1/p) = (q_2/p) = +1\), then

\[
\left( \frac{\varepsilon_{q_1 q_2}}{p} \right) = \left( \frac{p}{q_1} \right)_4 \left( \frac{q_1}{p} \right)_4 \left( \frac{p}{q_2} \right)_4 \left( \frac{q_2}{p} \right)_4.
\]

(Furuta [7: Corollary, p. 143])

**Proof.** As \((q_1/p) = (q_2/p) = 1\), we have \((q_1 q_2/p)_4 = (q_1/p)_4 (q_2/p)_4\), and by the law of quadratic reciprocity \((p/q_1) = (p/q_2) = 1\), so \([p/\pi_1]_4 = (p/q_1)_4\), \([p/\pi_2]_4 = (p/q_2)_4\). The result now follows immediately from Theorem 1.

**Corollary 2.** If \( p, q_1, q_2 \) are distinct primes congruent to 1 modulo 4, such that \((q_1/p) = (q_2/p) = -1\), \((q_1/q_2) = -1\), then

\[
\left( \frac{\varepsilon_{q_1 q_2}}{p} \right) = - \left( \frac{pq_1}{q_2} \right)_4 \left( \frac{pq_2}{q_1} \right)_4 \left( \frac{q_1 q_2}{p} \right)_4.
\]

**Proof.** As \((q_2/q_1) = -1\), we have \([q_2/\pi_1]_2 = -1\), that is, \([\pi_2/\pi_1]_2 [\pi_2/\pi_1]_2 = -1\).

Now, from \( T + i = i\pi_1 \pi_2 \alpha^2 \), we have
EVALUATION OF \( (\varepsilon_{q_1q_2}/p) \)

\[ 2 = \pi_1 \bar{\pi}_2 \alpha^2 + \bar{\pi}_1 \pi_2 \alpha^2, \]

so

\[ 2 \equiv \pi_1 \bar{\pi}_2 \alpha^2 \pmod{\pi_1}, \]

giving

\[ [2/\pi_1]_2 = [\pi_1/\pi_1][\bar{\pi}_2/\pi_1]_2 = [2/\pi_1][\bar{\pi}_2/\pi_1]_2, \]

that is,

\[ [\bar{\pi}_2/\pi_1]_2 = +1, [\pi_2/\pi_1]_2 = -1. \]

Hence we have

\[ [\pi_1/\pi_2]_4 [\pi_2/\pi_1]_4 = [\pi_1/\pi_2]_4 [\bar{\pi}_2/\pi_1]_4 = [\pi_1/\pi_2]_4 [\bar{\pi}_2/\pi_1]_4 \]

\[ = [\pi_1/\pi_2]_4 [\pi_2/\pi_1]_2 [\bar{\pi}_2/\pi_1]_4 \]

\[ = -[\pi_1/\pi_2]_4 [\pi_2/\pi_1]_4, \]

that is,

\[ (3) \]

\[ \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} \begin{bmatrix} \pi_2 \\ \pi_1 \end{bmatrix}_4 = -(-1)^{q_1-1} \cdot q_2^{-1}, \]

by the law of biquadratic reciprocity in \( \mathbb{Z}[i] \). Also, by the law of biquadratic reciprocity in \( \mathbb{Z}[i] \), we have

\[ (4) \]

\[ \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} \begin{bmatrix} \pi_2 \\ \pi_1 \end{bmatrix}_4 = (-1)^{q_1-1} \cdot q_2^{-1}. \]

Multiplying (3) and (4) together, we obtain

\[ [q_1/\pi_2]_4 [q_2/\pi_1]_4 = -1, \]

and Theorem 1 gives

\[ (\varepsilon_{q_1q_2}/p) = [p/\pi_1]_4 [p/\pi_2]_4 (q_1q_2/p)_4 \]

\[ = -[pq_1/\pi_2]_4 [pq_2/\pi_1]_4 (q_1q_2/p)_4, \]

\[ = -([pq_1/\pi_2]_4 [pq_2/\pi_1]_4 (q_1q_2/p)_4, \]

as required.

We are now in a position to obtain the explicit evaluation of \( (\varepsilon_{q_1q_2}/p) \), when \( p^l \) is represented by an ambiguous form of discriminant \( -4q_1q_2 \).

This is done, following ideas of Lehmer \[11: \text{pp.} \ 369-371\], by using the representation of \( p^l \) to compute the residue symbols appearing in the expression for \( (\varepsilon_{q_1q_2}/p) \) given in Theorem 1 or its corollaries. Many of the details are suppressed, as the calculations parallel those given by Lehmer. As in Lehmer's work, we require that \( l \) be odd, and an assumption to this effect is made wherever necessary. The results, which constitute Theorem 2, are given in the Table.
<table>
<thead>
<tr>
<th>Case</th>
<th>Assumptions</th>
<th>Evaluation of $\left( \frac{\epsilon_m}{p} \right)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$q_1 \equiv q_2 \equiv 1 \mod 8$</td>
<td>$\left( -\frac{1}{p} \right) = \left( \frac{q_1}{p} \right) = \left( \frac{q_2}{p} \right) = 1$ if $p^{h/16} = x^2 + q_1 q_2 y^2$ or $q_1 x^2 + q_2 y^2$</td>
</tr>
<tr>
<td></td>
<td>$q_1 = +1$</td>
<td>$N(\epsilon_m) = -1$ if $2p^{h/16} = x^2 + q_1 q_2 y^2$ or $q_1 x^2 + q_2 y^2$</td>
</tr>
<tr>
<td></td>
<td>$h \equiv 16 \mod 32$</td>
<td></td>
</tr>
<tr>
<td>II</td>
<td>$q_1 \equiv q_2 \equiv 1 \mod 8$</td>
<td>$\left( -\frac{1}{p} \right) = \left( \frac{q_1}{p} \right) = \left( \frac{q_2}{p} \right) = 1$ if $p^{h/8} = x^2 + q_1 q_2 y^2$</td>
</tr>
<tr>
<td></td>
<td>$q_1 = -1$</td>
<td>$-1$ if $2p^{h/8} = x^2 + q_1 q_2 y^2$</td>
</tr>
<tr>
<td></td>
<td>$h \equiv 8 \mod 16$</td>
<td></td>
</tr>
<tr>
<td>III</td>
<td>$q_1 \equiv 1, q_2 \equiv 5 \mod 8$</td>
<td>$\left( -\frac{1}{p} \right) = \left( \frac{q_1}{p} \right) = \left( \frac{q_2}{p} \right) = 1$ if $p^{h/8} = x^2 + q_1 q_2 y^2$</td>
</tr>
<tr>
<td></td>
<td>$q_1 = +1$</td>
<td>$N(\epsilon_m) = -1$ if $2p^{h/8} = x^2 + q_1 q_2 y^2$</td>
</tr>
<tr>
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<td>$\left( -\frac{1}{p} \right) = \left( \frac{q_1}{p} \right) = \left( \frac{q_2}{p} \right) = 1$ if $p^{h/8} = x^2 + q_1 q_2 y^2$</td>
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<td>$N(\epsilon_m) = -1$ if $2p^{h/8} = x^2 + q_1 q_2 y^2$</td>
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<tr>
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</tbody>
</table>

N.B. $T$ is defined by $\epsilon_m = T + U \sqrt{m}$, $\lambda = 1$ or $3$
$h$ is the classnumber of $Q(\sqrt{-m})$.
All representations are primitive.
Let I, A, B, C denote the classes of the forms \([1, 0, q_1q_2], [2, 2, \frac{1}{2}(q_1q_2 + 1)], [q_1, 0, q_2], [2q_1, 2q_1, \frac{1}{2}(q_1 + q_2)]\) respectively. These are precisely the ambiguous classes of forms of discriminant \(-4q_1q_2\), so that the classes of forms of discriminant \(-4q_1q_2\) fall into 4 genera. The generic characters are \(\chi_1(k) = (-1/k), \chi_2(k) = (k/q_1), \chi_2(k) = (k/q_2) (k > 0)\). The six cases appearing in the table are treated below.

**Case I.** \(q_1 \equiv q_2 \equiv 1 \pmod{8}\), \((q_1/q_2) = +1\). In this case I, A, B, C are all in the principal genus, so that \(h = h(-q_1q_2) \equiv 0 \pmod{16}\) (Brown [4: Theorem 1]). Thus, if \(p\) is a prime, such that \((-1/p) = (q_1/p) = (q_2/p) = 1\), there are positive coprime integers \(x\) and \(y\) such that \(p^i = x^2 + q_1q_2y^2, 2x^2 + 2xy + \frac{1}{2}(q_1q_2 + 1)y^2, q_1x^2 + q_2y^2, 2q_1x^2 + 2q_1xy + \frac{1}{2}(q_1 + q_2)y^2\); that is, there are positive coprime integers \(x\) and \(y\) such that

\[ p^i \text{ or } 2p^i = x^2 + q_1q_2y^2 \text{ or } q_1x^2 + q_2y^2, \]

where \(l = h/16\). We now assume that \(N(\varepsilon_{q_1q_2}) = -1\) and \(h \equiv 16 \pmod{32}\) (so that \(l\) is odd). These are two independent assumptions since: \(N(\varepsilon_{17, 89}) = -1\) and \(h(-41, 241) = 112 \equiv 16 \pmod{32}\), whereas \(N(\varepsilon_{17, 89}) = +1\) and \(h(-17, 89) = 16\); also \(N(\varepsilon_{17, 281}) = -1\) and \(h(-17, 281) = 32\), whereas \(N(\varepsilon_{17, 137}) = +1\) and \(h(-17, 137) = 32\).

Taking \(p^i = x^2 + q_1q_2y^2\) modulo \(p\), \(q_1\) and \(q_2\), we obtain

\[(q_1/p)_{4}(q_2/p)_{4} = (2/p)(x/p)(y/p),\]

\[(p/q_1)_{4} = (x/q_1), (p/q_2)_{4} = (x/q_2),\]

so that, by Corollary 1, we have

\[
\left( \frac{\varepsilon_{q_1q_2}}{p} \right) = \left( \frac{2}{p} \right) \left( \frac{x}{p} \right) \left( \frac{y}{p} \right) \left( \frac{x}{q_1} \right) \left( \frac{x}{q_2} \right).
\]

Next we set

\[
x_1 = 2^\alpha x_1, x_1 \equiv 1 \pmod{2}, \alpha \geq 0,\]

\[
y_1 = 2^\beta y_1, y_1 \equiv 1 \pmod{2}, \beta \geq 0.
\]

By the law of quadratic reciprocity, we have (as \(l\) is odd)

\[
(x/p) = (2/p)^\alpha(x_1/p) = (2/p)^\alpha(p/x_1) = (2/p)^\alpha(p^i/x_1) = (2/p)^\alpha(q_1/x_1)(q_2/x_1),
\]

\[
(y/p) = (2/p)^\beta(y_1/p) = (2/p)^\beta(p/y_1) = (2/p)^\beta(p^i/y_1) = (2/p)^\beta,
\]

\[
(x/q_1) = (2/q_1)^\alpha(x_1/q_1) = (x_1/q_1), (x/q_2) = (2/q_2)^\alpha(x_1/q_2) = (x_1/q_2),
\]

giving

\[
\left( \frac{\varepsilon_{q_1q_2}}{p} \right) = \left( \frac{2}{p} \right)^{1+\alpha+\beta}.
\]
If \( p \equiv 1 \pmod{8} \), \( (2/p) = +1 \), so \( (\varepsilon_{q_1|q_2}/p) = +1 \); if \( p \equiv 5 \pmod{8} \), then \( \alpha + \beta = 1 \), and again \( (\varepsilon_{q_1|q_2}/p) = +1 \).

Similarly, using \( p' = q_1x^2 + q_2y^2 \) in Corollary 1, we obtain

\[
\left( \frac{\varepsilon_{q_1|q_2}}{p} \right) = \left( \frac{q_1}{q_2} \right)_4 \left( \frac{q_2}{q_1} \right)_4.
\]

But, as \( N(\varepsilon_{q_1|q_2}) = -1 \), we have \( (q_1/q_2)_4(q_2/q_1)_4 = +1 \) (Brown [2: Lemma 4]), so that \( (\varepsilon_{q_1|q_2}/p) = +1 \).

Using \( 2p' = x^2 + q_1q_2y^2 \) in Corollary 1, we obtain, using the easily proved result \( (2/p)(2/x)(2/y) = (1)^{(q_1+q_2-2)/8} \),

\[
\left( \frac{\varepsilon_{q_1|q_2}}{p} \right) = ( -1 )^{(q_1+q_2-2)/8} \left( \frac{2}{q_1} \right)_4 \left( \frac{2}{q_2} \right)_4 = \left( \frac{e}{q_1} \right)_4 \left( \frac{e}{q_2} \right)_4 = -1,
\]

where \( d, e \) are positive odd integers defined by \( q_1q_2 = 2e^2 - d^2 \). As \( (q_1/q_2)_4(q_2/q_1)_4 = +1 \) (since \( N(\varepsilon_{q_1|q_2}) = -1 \)) and \( h(-q_1q_2) \equiv 16 \pmod{32} \), we have \( (e/q_1)(e/q_2) = -1 \) (Kaplan [9: Prop. C]), so that \( (\varepsilon_{q_1|q_2}/p) = -1 \).

Using \( 2p' = q_1x^2 + q_2y^2 \) in Corollary 1, we obtain in a similar manner

\[
\left( \frac{\varepsilon_{q_1|q_2}}{p} \right) = ( -1 )^{(q_1+q_2-2)/8} \left( \frac{2}{q_1} \right)_4 \left( \frac{2}{q_2} \right)_4 \left( \frac{q_1}{q_2} \right)_4 \left( \frac{q_2}{q_1} \right)_4 = \left( \frac{e}{q_1} \right)_4 \left( \frac{e}{q_2} \right)_4 = -1,
\]

CASE II. \( q_1 = q_2 \equiv 1 \pmod{8} \), \( (q_1/q_2) = -1 \). In this case I, A are in the principal genus and B, C are in the non-principal genus for which \( \chi_1 = +1 \), so that \( h = h(-q_1q_2) \equiv 0 \pmod{8} \) (Brown [4: Theorem 1]). Thus, if \( p \) is a prime such that \( (-1/p) = (q_1/p) = (q_2/p) = 1 \), there are positive coprime integers \( x \) and \( y \) such that

\[
p' \text{ or } 2p' = x^2 + q_1q_2y^2,
\]

where \( l = h/8 \), and, if \( (-1/p) = 1 \), \( (q_1/p) = (q_2/p) = -1 \), such that

\[
p' \text{ or } 2p' = q_1x^2 + q_2y^2.
\]

As \( (q_1/q_2) = -1 \) we have \( N(\varepsilon_{q_1|q_2}) = -1 \) (Dirichlet [6: p. 228]), and we assume that \( h \equiv 8 \pmod{16} \) (so that \( l \) is odd). The example \( q_1 = 17, q_2 = 73, h = h(-1241) = 32 \), shows that this is a genuine assumption.

Using \( p' = x^2 + q_1q_2y^2 \) in Corollary 1 we obtain \( (\varepsilon_{q_1|q_2}/p) = +1 \).

Using \( 2p' = x^2 + q_1q_2y^2 \) in Corollary 1, we obtain

\[
\left( \frac{\varepsilon_{q_1|q_2}}{p} \right) = ( -1 )^{(q_1+q_2-2)/8} \left( \frac{2}{q_1} \right)_4 \left( \frac{2}{q_2} \right)_4,
\]

the right hand side of which is \(-1\), as \( h(-q_1q_2) \equiv 8 \pmod{16} \) (Kaplan [9: Prop. B]).
EVALUATION OF $(\epsilon_{q_1,q_2}/p)$

Using $p^i = q_1 x^2 + q_2 y^2$ in Corollary 2 we obtain $(\epsilon_{q_1,q_2}/p) = +1$.

Finally, using $2p^i = q_1 x^2 + q_2 y^2$ in Corollary 2, we obtain

\[ \left( \frac{\epsilon_{q_1,q_2}}{p} \right) = \left( \frac{-1}{q_1} \right)^{\frac{q_1+q_2-2}{8}} \left( \frac{2}{q_2} \right) \left( \frac{2}{q_1} \right), \]

the right hand side of which is $-1$, as $h(-q_1q_2) \equiv 8 \pmod{16}$ (Kaplan [9: Prop. B]).

**Case III.** $q_1 \equiv 1, \, q_2 \equiv 5 \pmod{8}, \, (q_1/q_2) = +1$. In this case I, B are in the principal genus and A, C are in a non-principal genus for which $\chi_1 = -1$. We have $h = h(-q_1q_2) \equiv 0 \pmod{8}$ (Brown [4: Theorem 1]). Thus, if $p$ is a prime for which $(-1/p) = (q_1/p) = (q_2/p) = +1$, there are positive coprime integers $x$ and $y$ such that

\[ p^i = x^2 + q_1q_2 y^2 \text{ or } q_1x^2 + q_2 y^2, \]

where $l = h/8$. We now assume that $N(\epsilon_{q_1,q_2}) = -1$ and $h \equiv 8 \pmod{16}$ (so that $l$ is odd).

These are two independent assumptions since: $N(\epsilon_{17,53}) = -1$ and $h(-17 \cdot 53) = 24 \equiv 8 \pmod{16}$, whereas $N(\epsilon_{17,229}) = +1$ and $h(-17 \cdot 229) = 40 \equiv 8 \pmod{16}$; also $N(\epsilon_{1601,5}) = -1$ and $h(-1601 \cdot 5) = 48 \equiv 0 \pmod{16}$, whereas $N(\epsilon_{17,13}) = +1$ and $h(-17 \cdot 13) = 16 \equiv 0 \pmod{16}$.

Using $p^i = x^2 + q_1q_2 y^2$ in Corollary 1, we obtain $(\epsilon_{q_1,q_2}/p) = (-1)^y$. Using $p^i = q_1 x^2 + q_2 y^2$ in Corollary 1, we obtain

\[ \left( \frac{\epsilon_{q_1,q_2}}{p} \right) = (-1)^y \left( \frac{q_1}{q_2} \right) \left( \frac{q_2}{q_1} \right). \]

As $N(\epsilon_{q_1,q_2}) = -1$, we have $(q_1/q_2)(q_2/q_1)_4 = +1$ (Brown [2: Lemma 4]), so that $(\epsilon_{q_1,q_2}/p) = (-1)^y$.

**Case IV.** $q_1 \equiv 1, \, q_2 \equiv 5 \pmod{8}, \, (q_1/q_2) = -1$. In this case I, A, B, C are each in different genera, with I in the principal genus and B in the non-principal genus with $\chi_1 = +1$. We have $h = h(-q_1q_2) \equiv 4 \pmod{8}$ (Brown [4: Theorem 1]). Thus, if $p$ is a prime such that $(-1/p) = (q_1/p) = (q_2/p) = 1$, there exist positive coprime integers $x$ and $y$ such that $p^i = x^2 + q_1q_2 y^2$, where $l = h/4$ is odd, and such that $p^i = q_1 x^2 + q_2 y^2$, if $(-1/p) = 1$, $(q_1/p) = (q_2/p) = -1$. As $(q_1/q_2) = -1$, a theorem of Dirichlet [6: p. 228] guarantees that $N(\epsilon_{q_1,q_2}) = -1$. Using $p^i = x^2 + q_1q_2 y^2$ in Corollary 1, we obtain $(\epsilon_{q_1,q_2}/p) = (-1)^y$, and using $p^i = q_1 x^2 + q_2 y^2$ in Corollary 2, we also obtain $(\epsilon_{q_1,q_2}/p) = (-1)^y$.

**Case V.** $q_1 \equiv q_2 \equiv 5 \pmod{8}, \, (q_1/q_2) = +1$. In this case I, B are in the principal genus and A, C are in the non-principal genus with $\chi_1 = +1$. We have $h = h(-q_1q_2) \equiv 0 \pmod{8}$ (Brown [4: Theorem 1]). Thus, if $p$
is a prime such that \((-1/p) = (q_1/p) = (q_2/p) = 1\), there exist positive coprime integers \(x\) and \(y\) such that \(p' = x^2 + q_1q_2y^2\) or \(q_1x^2 + q_2y^2\); and, if \((-1/p) = 1, (q_1/p) = (q_2/p) = -1\), such that \(2p' = x^2 + q_1q_2y^2\) or \(q_1x^2 + q_2y^2\), where \(l = h/8\). We assume that \(N(\epsilon_{q_1q_2}) = -1\), so that by a theorem of Brown [2: Lemma 4] we have \((q_1/q_2)_4(2p/q_2)_4(2p/q_1)_4 = 1\), and hence by a theorem of Kaplan [9: Prop. B1] we have \(h \equiv 8 \pmod{16}\), so that \(l\) is odd. Using \(p' = x^2 + q_1q_2y^2\) in Corollary 1, we obtain \((\epsilon_{q_1q_2}/p) = +1\), and using \(p' = q_1x^2 + q_2y^2\) in the same corollary we obtain \((\epsilon_{q_1q_2}/p) = -(q_1/q_2)_4(q_2/q_1)_4 = -1\).

When \(2p' = x^2 + q_1q_2y^2\) or \(q_1x^2 + q_2y^2\) the evaluation of \((\epsilon_{q_1q_2}/p)\) appears to be more difficult. It was originally hoped to give a third corollary to Theorem 1 expressing \((\epsilon_{q_1q_2}/p)\) in terms of \((2p/q_1)_4(2p/q_2)_4(q_1q_2/p)_4\) when \(p, q_1, q_2\) are distinct primes congruent to 1 modulo 4, and such that \((q_1/p) = (q_2/p) = -1, (q_1/q_2) = +1, q_1 \equiv q_2 \equiv 5 \pmod{8}\). No such representation was found, and so instead we apply Theorem 1 directly.

If \(2p' = x^2 + q_1q_2y^2\) we have

\[
\left( \frac{\epsilon_{q_1q_2}}{p} \right) = \left( \frac{p}{\pi_1} \right)_4 \left( \frac{p}{\pi_2} \right)_4 \left( \frac{q_1q_2}{p} \right)_4
\]

\[
\left( \frac{2}{\pi_1} \right)_4 \left( \frac{2}{\pi_2} \right)_4 \left( \frac{2p'}{\pi_1} \right)_4 \left( \frac{2p'}{\pi_2} \right)_4 \left( \frac{q_1q_2}{p} \right)_4
\]

\[
\left( \frac{2}{\pi_1} \right)_4 \left( \frac{2}{\pi_2} \right)_4 \left( \frac{x}{q_1} \right)_4 \left( \frac{x}{q_2} \right)_4 \left( \frac{y}{p} \right)_4 \left( \frac{y}{p} \right)_4
\]

\[
\left( \frac{2}{q_1} \right) = \left( \frac{2}{q_2} \right) = \left( \frac{x}{q_1} \right)_4 \left( \frac{x}{q_2} \right)_4 = -1, \left( \frac{q_1q_2}{p} \right)_4 = \left( \frac{2}{p} \right)_4 \left( \frac{x}{p} \right)_4 \left( \frac{y}{p} \right)_4.
\]

Now, by Jacobi’s form of the law of quadratic reciprocity, we have (as \(l\) is odd)

\[
\left( \frac{x}{p} \right) = \left( \frac{p}{x} \right) \left( \frac{2p'}{x} \right) = \left( \frac{x}{q_1} \right)_4 \left( \frac{x}{q_2} \right)_4 \left( \frac{2p'}{x} \right)_4 \left( \frac{x}{q_1} \right)_4 \left( \frac{x}{q_2} \right)_4 \left( \frac{2p'}{x} \right)_4
\]

\[
\left( \frac{y}{p} \right) = \left( \frac{p}{y} \right) \left( \frac{2p'}{y} \right) = \left( \frac{y}{q_1} \right)_4 \left( \frac{y}{q_2} \right)_4 \left( \frac{2p'}{y} \right)_4 \left( \frac{y}{q_1} \right)_4 \left( \frac{y}{q_2} \right)_4 \left( \frac{2p'}{y} \right)_4
\]

so

\[
\left( \frac{\epsilon_{q_1q_2}}{p} \right) = \left( \frac{2}{\pi_1} \right)_4 \left( \frac{2}{\pi_2} \right)_4 \left( \frac{2}{p} \right)_4 \left( \frac{x}{y} \right)_4 = \left( -1 \right)^{(q_1+q_2-2)/8} \left( \frac{2}{\pi_1\pi_2} \right)_4
\]

Setting \(\alpha = g + hi\), where \(\alpha\) is defined by (1), we have
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\[
\left( \frac{\varepsilon_{q_1q_2}}{p} \right) = (-1)^{(q_1+q_2-2)/8} \left[ \frac{2}{\pi_1 \pi_2 \alpha^2} \right]_{A \alpha_2}^{[2]} \left[ \frac{2}{1 - Ti} \right]_{A \{g + hi\}} \text{ (by (1))}
\]

\[
= (-1)^{(q_1+q_2-2)/8 + T/4 + h/2},
\]

by the supplements to the laws of quadratic and biquadratic reciprocity in $\mathbb{Z}[i]$, since $T \equiv 0 \pmod{4}$ in this case. As $\pi_j (j = 1, 2)$ is a primary prime factor of $q_j (j = 1, 2)$, we have $\pi_j = a_j + ib_j$, $a_j \equiv 1 \pmod{2}$, $b_j \equiv 0 \pmod{2}$, $a_j + b_j - 1 \equiv 0 \pmod{4}$, $a_j^2 + b_j^2 = q_j$. Since $q_j \equiv 5 \pmod{8}$, we have, for $j = 1, 2$,

\[
\begin{cases}
    a_j \equiv 7 \pmod{8}, & b_j \equiv 2 \pmod{4}, \quad \text{if } q_j \equiv 5 \pmod{16}, \\
    a_j \equiv 3 \pmod{8}, & b_j \equiv 2 \pmod{4}, \quad \text{if } q_j \equiv 13 \pmod{16}.
\end{cases}
\]

Set $a + ib = \pi_1 \pi_2$, so we have

\[
a = a_1 a_2 - b_1 b_2, \quad b = a_1 b_2 + a_2 b_1.
\]

Clearly we have

\[
a \equiv 5 \pmod{8}, \quad b \equiv 0 \pmod{4}, \quad \text{if } q_1 + q_2 \equiv 10 \pmod{16}, \\
a \equiv 1 \pmod{8}, \quad b \equiv 0 \pmod{4}, \quad \text{if } q_1 + q_2 \equiv 2 \pmod{16}.
\]

From $1 - Ti = \pi_1 \pi_2 \alpha^2 = (a + ib)(g + ih)^2$, we have

\[
1 = a(g^2 - h^2) - b(2gh),
\]

so that

\[
g \equiv 1 \pmod{2}, \quad h \equiv 2 \pmod{4}, \quad \text{if } q_1 + q_2 \equiv 10 \pmod{16}, \\
g \equiv 1 \pmod{2}, \quad h \equiv 0 \pmod{4}, \quad \text{if } q_1 + q_2 \equiv 2 \pmod{16},
\]

giving

\[
h/2 \equiv (q_1 + q_2 - 2)/8 \pmod{2},
\]

so that

\[
(\varepsilon_{q_1q_2}/p) = (-1)^{T/4}.
\]

Similarly one can prove that $(\varepsilon_{q_1q_2}/p) = (-1)^{T/4 + 1}$, when $2p^l = q_1 x^2 + q_2 y^2$, using $(q_1/q_2)_4(q_2/q_1)_4 = +1$.

**Case VI.** $q_1 \equiv q_2 \equiv 5 \pmod{8}$, $(q_1/q_2) = -1$. In this case I and C are in the principal genus and A and B are in the non-principal genus with $\chi_1 = +1$. We have $h = h(-q_1q_2) \equiv 0 \pmod{8}$ (Brown [4: Theorem 1]). Thus, if $p$ is a prime such that $(-1/p) = (q_1/p) = (q_2/p) = +1$, there are positive coprime integers $x$ and $y$ such that $p^l = x^2 + q_1q_2 y^2$ or $2p^l = \ldots$
and if \((-1/p) = 1\), \((q_1/p) = (q_2/p) = -1\), such that \(p^i = q_1 x^2 + q_2 y^2\) or \(2p^i = x^2 + q_1 q_2 y^2\), where \(l = h/8\). As \((q_1/q_2) = -1\), by Dirichlet's theorem \([6]: p. 228\), we have \(N(e_{q_1q_2}) = -1\), and we assume that \(h \equiv 8 \pmod{16}\), so that \(l\) is odd. The example \(q_1 = 5, q_2 = 37, h = h(-185) = 16\), shows that this is a genuine assumption.

Using \(p^i = x^2 + q_1 q_2 y^2\) in Corollary 1, we obtain \((e_{q_1q_2}/p) = +1\), and using \(2p^i = q_1 x^2 + q_2 y^2\) in Corollary 1, we obtain

\[
\left( \frac{e_{q_1q_2}}{p} \right) = (-1)^{(q_1 + q_2 - 2)/8} \left( \frac{2q_1}{q_2} \right) \left( \frac{2q_2}{q_1} \right),
\]

the right hand side of which is \(-1\), as \(h \equiv 8 \pmod{16}\) (Kaplan \([9]: Prop. B'\)). Using \(2p^i = x^2 + q_1 q_2 y^2\) in Corollary 2, we obtain

\[
\left( \frac{e_{q_1q_2}}{p} \right) = (-1)^{(q_1 + q_2 + 6)/8} \left( \frac{2q_1}{q_2} \right) \left( \frac{2q_2}{q_1} \right) = +1.
\]

Finally using \(p^i = q_1 x^2 + q_2 y^2\) in Corollary 2, we obtain \((e_{q_1q_2}/p) = -1\).

This completes the proof of Theorem 2. We remark that parts of II and VI of Theorem 2 have been proved without the restriction \(h(-q_1q_2) \equiv 8 \pmod{16}\) using class field theory \([5]\).

We conclude with a few examples to illustrate the theorem.

EXAMPLE 1. (Compare Kuroda \([10]: pp. 155-156\)) Choose \(q_1 = 5, q_2 = 13\), so that \((q_1/q_2) = -1\), and \(h = h(-q_1q_2) = h(-65) = 8\). By part VI of Theorem 2, if \(p\) is a prime such that

\[
\left( \frac{-1}{p} \right) = \left( \frac{5}{p} \right) = \left( \frac{13}{p} \right) = +1,
\]

then

\[
\left( \frac{e_{65}}{p} \right) = \left( \frac{8 + \sqrt{65}}{p} \right) = \begin{cases} +1, & \text{if } p = x^2 + 65y^2, \\ -1, & \text{if } 2p = 5x^2 + 13y^2; \end{cases}
\]

and if \(p\) is such that

\[
\left( \frac{-1}{p} \right) = +1, \left( \frac{5}{p} \right) = \left( \frac{13}{p} \right) = -1,
\]

then

\[
\left( \frac{e_{65}}{p} \right) = \left( \frac{8 + \sqrt{65}}{p} \right) = \begin{cases} +1, & \text{if } 2p = x^2 + 65y^2, \\ -1, & \text{if } p = 5x^2 + 13y^2. \end{cases}
\]

Thus, for example, we have

\[
\left( \frac{e_{65}}{601} \right) = +1, \quad \text{as } 601 = 4^2 + 65 \cdot 3^2,
\]
EVALUATION OF \((\varepsilon_{q_1 q_2}/p)\)

\[
\left(\frac{\varepsilon_{65}}{29}\right) = -1, \quad \text{as } 2 \cdot 29 = 5 \cdot 3^2 + 13 \cdot 1^2,
\]

\[
\left(\frac{\varepsilon_{65}}{37}\right) = +1, \quad \text{as } 2 \cdot 37 = 3^2 + 65 \cdot 1^2,
\]

\[
\left(\frac{\varepsilon_{65}}{193}\right) = -1, \quad \text{as } 193 = 5 \cdot 6^2 + 13 \cdot 1^2.
\]

These are easily verified directly:

\[
\left(\frac{\varepsilon_{65}}{601}\right) = \left(\frac{8 + 234}{601}\right) = \left(\frac{242}{601}\right) = \left(\frac{2}{601}\right) = +1,
\]

\[
\left(\frac{\varepsilon_{65}}{29}\right) = \left(\frac{8 + 6}{29}\right) = \left(\frac{14}{29}\right) = -1,
\]

\[
\left(\frac{\varepsilon_{65}}{37}\right) = \left(\frac{8 + 18}{37}\right) = \left(\frac{26}{37}\right) = +1,
\]

\[
\left(\frac{\varepsilon_{65}}{193}\right) = \left(\frac{8 + 114}{193}\right) = \left(\frac{122}{193}\right) = -1.
\]

**Example 2.** Choose \(q_1 = 5\), \(q_2 = 29\), so that \((q_1/q_2) = +1\), \(N(\varepsilon_{q_1 q_2}) = N(\varepsilon_{145}) = N(12 + \sqrt{145}) = -1\), \(h = h(-q_1 q_2) = h(-145) = 8\). By part V of Theorem 2, we have

\[
\left(\frac{-1}{p}\right) = \left(\frac{5}{p}\right) = \left(\frac{29}{p}\right) = +1,
\]

then

\[
\left(\frac{\varepsilon_{145}}{p}\right) = \left(\frac{12 + \sqrt{145}}{p}\right) = \begin{cases} +1, & \text{if } p = x^2 + 145y^2, \\ -1, & \text{if } p = 5x^2 + 29y^2; \end{cases}
\]

and if \(p\) is such that

\[
\left(\frac{-1}{p}\right) = +1, \quad \left(\frac{5}{p}\right) = \left(\frac{29}{p}\right) = -1,
\]

then

\[
\left(\frac{\varepsilon_{145}}{p}\right) = \left(\frac{12 + \sqrt{145}}{p}\right) = \begin{cases} +1, & \text{if } 2p = 5x^2 + 29y^2, \\ -1, & \text{if } 2p = x^2 + 145y^2. \end{cases}
\]

**Example 3.** Choose \(q_1 = 17\), \(q_2 = 5\), so that \((q_1/q_2) = -1\), and \(h = (-q_1 q_2) = h(-85) = 4\). By part IV of Theorem 2, we have that if \(p\) is a prime such that

\[
\left(\frac{-1}{p}\right) = \left(\frac{85}{p}\right) = +1,
\]

then
Thus, for example, we have
\[
\left( \frac{\varepsilon_{85}}{p} \right) = \left( \frac{\sqrt{85}}{p} \right) = \begin{cases} (-1)^y, & \text{if } \left( \frac{17}{p} \right) = \left( \frac{5}{p} \right) = 1, \quad p = x^2 + 85y^2, \\ (-1)^y, & \text{if } \left( \frac{17}{p} \right) = \left( \frac{5}{p} \right) = -1, \quad p = 17x^2 + 5y^2. \end{cases}
\]

Thus, for example, we have
\[
\left( \frac{\varepsilon_{85}}{349} \right) = \left( \frac{\sqrt{85} + 145}{349} \right) = \left( \frac{77}{349} \right) = +1, \quad 349 = 3^2 + 85 \cdot 2^2,
\]
\[
\left( \frac{\varepsilon_{85}}{89} \right) = \left( \frac{\sqrt{85} + 21}{89} \right) = \left( \frac{15}{89} \right) = -1, \quad 89 = 2^3 + 85 \cdot 1^2,
\]
\[
\left( \frac{\varepsilon_{85}}{37} \right) = \left( \frac{\sqrt{85} + 23}{37} \right) = \left( \frac{16}{37} \right) = +1, \quad 37 = 17 \cdot 1^2 + 5 \cdot 2^2,
\]
\[
\left( \frac{\varepsilon_{85}}{73} \right) = \left( \frac{\sqrt{85} + 31}{73} \right) = \left( \frac{20}{73} \right) = -1, \quad 73 = 17 \cdot 2^2 + 5 \cdot 1^2.
\]

**Example 4.** Choose \( q_1 = 17 \), \( q_2 = 53 \), so that \( (q_1/q_2) = +1 \), \( h = h(-q_1q_2) = h(-901) = 24 \), \( N(\varepsilon_{q_1q_2}) = N(\varepsilon_{901}) = -1 \). By part III of Theorem 2, we have that if \( p \) is a prime such that
\[
\left( \frac{-1}{p} \right) = \left( \frac{17}{p} \right) = \left( \frac{53}{p} \right) = +1,
\]
then
\[
\left( \frac{\varepsilon_{901}}{p} \right) = \left( \frac{30 + \sqrt{901}}{p} \right) = (-1)^y,
\]
where
\[
p^3 = x^2 + 901y^2 \text{ or } p^3 = 17x^2 + 53y^2.
\]

Thus, for example, we have
\[
\left( \frac{\varepsilon_{901}}{89} \right) = \left( \frac{30 + 79}{89} \right) = \left( \frac{5}{89} \right) = +1, \quad 89^3 = 587^2 + 901 \cdot 20^2,
\]
\[
\left( \frac{\varepsilon_{901}}{13} \right) = \left( \frac{30 + 2}{13} \right) = \left( \frac{2}{13} \right) = -1, \quad 13^3 = 36^2 + 901 \cdot 1^2,
\]
\[
\left( \frac{\varepsilon_{901}}{149} \right) = \left( \frac{30 + 93}{149} \right) = \left( \frac{123}{149} \right) = +1, \quad 149^3 = 17 \cdot 269^2 + 53 \cdot 198^2,
\]
\[
\left( \frac{\varepsilon_{901}}{1753} \right) = \left( \frac{30 + 253}{1753} \right) = \left( \frac{283}{1753} \right) = -1, \quad 1753^3 = 17 \cdot 15410^2 + 53 \cdot 5047^2.
\]

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