On the least quadratic non-residue of a prime
\[ p \equiv 3 \pmod{4} \]

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Dedicated to Alfred Brauer on his 86th birthday

1. Introduction and summary

The problem of finding an upper bound for the least quadratic non-residue of an odd prime \( p \) is of historical interest because of a remark of Gauss [7] (see, for example, [3], p. 27), regarding its difficulty. Upper bounds of the order of \( p^{\frac{1}{3}} \) have been given by numerous authors including Vinogradov [21], Brauer and Reynolds [4], Kanold [13], Nagell [14], [15], [16], [17], Rédei [18], Skolem [19], and Hudson [10].

The well-known method of Vinogradov [20], used in conjunction with the character sum estimates of Burgess [5], [6], yields a sharp bound for the least quadratic non-residue of “sufficiently large” primes. Only one author, Brauer [1], has exhibited a purely combinatorial method for bounding the least quadratic non-residue of a prime which yields a bound that is \( o(p^{\frac{1}{3}}) \). In 1931, Brauer [1] showed that the smallest positive quadratic non-residue \( q \) of an odd prime \( \equiv 1 \pmod{8} \) must satisfy

\[
(1.1) \quad q < (2p)^{\frac{2}{3}} + 3(2p)^{\frac{1}{3}} + 1 \approx 1.3195 \, p^{\frac{2}{3}} + 3.446 \, p^{\frac{1}{3}} + 1.
\]

Using this method, Brauer [2], Whyburn [22], and Hudson [8], were able to obtain bounds for the second and third smallest prime \( k \)-th power non-residues in certain cases, and this method was used by Hudson [9], [11], [12] in related problems, for example, in providing upper bounds for the first three consecutive quadratic residues of a prime \( p > 17 \) and for the least \( k \)-th power non-residue \( \pmod{p} \) in an arithmetic progression.

Brauer informs us that several authors (unpublished) have been able to obtain slight improvements in the coefficient of \( p^{\frac{1}{3}} \) in (1.1), with easy refinements of his proof. This is, of course, not of great interest as it does not appreciably improve Brauer’s bound for large \( p \). In his classes, for more than 40 years, Brauer gave the proof of (1.1) and challenged his students to improve this bound. In this paper we give a method which yields a small improvement (approximately 24%). In particular, we prove in section 2, that if \( q \) is the least positive quadratic non-residue of an odd prime \( \equiv 1 \pmod{8} \), then

\[
(1.2) \quad q < p^{\frac{2}{3}} + 12p^{\frac{1}{3}} + 33.
\]

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2. A bound for the least non-residue of odd primes $p \equiv \not{1} \pmod{8}$

**Theorem.** Let $p$ be an odd prime $\equiv \not{1} \pmod{8}$ and let $q$ denote the least quadratic non-residue of $p$. Then

\begin{equation}
q < p^{5/8} + 12p^{3/8} + 33.
\end{equation}

**Proof.** Assume otherwise, so that

\begin{equation}
q > p^{5/8} + 12p^{3/8} + 33.
\end{equation}

Since $q > 2$ we may take $p$ to be $\equiv 7 \pmod{8}$. Also (2.2) implies that $p > 71$. Now 8 is a quadratic residue and $-1$ is a quadratic non-residue of $p$, so that the $q - 1$ positive integers

\begin{equation}
p - 8(q - 1), p - 8(q - 2), \ldots, p - 8
\end{equation}

are quadratic non-residues $\equiv 7 \pmod{8}$. (The integers in (2.3) are positive since

\[ q < p^{5/8} \quad \text{if} \quad p > 23, \]

see [3], p. 27.)

Let $r$ be an odd positive integer of the form

\begin{equation}
r = \left\lfloor p^{5/8} \right\rfloor + \alpha,
\end{equation}

where $\alpha$ is a positive integer $\leq 8$ to be chosen later.

Since

\begin{equation}
p^{5/8} < r \leq p^{5/8} + 8,
\end{equation}

we must have

\begin{equation}
r \leq q - 1
\end{equation}

in view of (2.2).

Let $h$ be the unique integer satisfying

\begin{equation}
8h \equiv 8q - p \pmod{r}, \quad 1 \leq h \leq r.
\end{equation}

By (2.7), we may define an integer $k$ by

\begin{equation}
k = \frac{p - 8(q - h)}{r}.
\end{equation}

From (2.6) and (2.7), we have $1 \leq h \leq q - 1$, so that the numerator in (2.8) is one of the integers in (2.3), and so $k$ is positive.

Now, set $l = \left\lfloor p^{3/8} \right\rfloor + 4$ so that

\begin{equation}
\end{equation}
Further, choose

(2. 10) \[ a = \left[ \frac{k}{2} \right] + 1. \]

Then \( \frac{1}{2} < a \leq \frac{k}{2} + 1 \) so that \((a - 1)^2 \leq k < a^2\).

Finally, choose \(\alpha\) such that

(2. 11) \[ r \equiv 1 \pmod{8}, \quad \text{if} \quad a \equiv 0 \quad \text{or} \quad 3 \pmod{4}, \]

and

(2. 12) \[ r \equiv 5 \pmod{8}, \quad \text{if} \quad a \equiv 1 \quad \text{or} \quad 2 \pmod{4}. \]

Provided that

(2. 13) \[ (k + 8l - 8) r \leq p - 8, \]

the integers \(kr, (k + 8)r, \ldots, (k + 8l - 8)r\) are in (2. 3) and so the \(l\) integers

(2. 14) \[ k, k + 8, \ldots, k + 8l - 8 \]

are all quadratic non-residues, which are \( \equiv 7 \pmod{8}\), if \(r \equiv 1 \pmod{8}\), and are \( \equiv 3 \pmod{8}\), if \(r \equiv 5 \pmod{8}\). The condition (2. 13) is satisfied as, by (2. 2), (2. 5), (2. 7), (2. 8), and (2. 9), we have

\[
(k + 8l - 8)r < p - 8q + 8r + 8r(p^{\frac{1}{2}} + 4) - 8r < p - 8q + 8(p^{\frac{1}{2}} + 8)(p^{\frac{1}{2}} + 4) < p - 8p^{\frac{2}{3}} - 96p^{\frac{1}{3}} - 264 + 8p^{\frac{1}{2}} + 96p^{\frac{1}{3}} + 256 = p - 8.
\]

If \(a\) is even, we consider the sequence of integers

(2. 15) \[ (a + 1) (a - 1), (a + 3) (a - 3), \ldots, (a + 2b - 1) (a - 2b + 1), \]

where \(b\) is the largest integer such that

(2. 16) \[ (a + 2b - 1) (a - 2b + 1) > (a - 1)^2; \]

if \(a\) is odd, we consider the sequence of integers

(2. 17) \[ (a + 2)(a) (a + 4) (a - 2), \ldots, (a + 2c) (a - 2c + 2), \]

where \(c\) is the largest integer such that

(2. 18) \[ (a + 2c) (a - 2c + 2) > (a - 1)^2. \]

The integers in (2. 15) are \( \equiv 7 \pmod{8}\), if \(a \equiv 0 \pmod{4}\), and are \( \equiv 3 \pmod{8}\), if \(a \equiv 2 \pmod{4}\). The integers in (2. 17) are \( \equiv 3 \pmod{8}\), if \(a \equiv 1 \pmod{4}\), and \( \equiv 7 \pmod{8}\), if \(a \equiv 3 \pmod{4}\). By the choices made in (2. 11) and (2. 12), we see that the integers in (2. 14) are in the same residue class modulo 8 as those in (2. 15), if \(a\) is even, and as those in (2. 17), if \(a\) is odd.

Next, we have \((a - 1)^2 \leq k < \frac{p}{r} < r^{\frac{4}{3}}\), so that \(a < p^{\frac{2}{3}} + 1\). Then

\[
a + 2b - 1 < a + \sqrt{2a - 1} < p^{\frac{2}{3}} + \sqrt{2(p^{\frac{2}{3}} + 1)} + 1 < q,
\]

so that the integers in (2. 15) are all quadratic residues. Similarly the integers in (2. 17) are also quadratic residues.
Thus, subdividing the integer interval
\[
\begin{cases}
[(a - 1)^2, \ldots, a^2 - 1], & \text{if } a \text{ is even}, \\
[(a - 1)^2, \ldots, a^2 + 2a], & \text{if } a \text{ is odd},
\end{cases}
\]
by the quadratic residues in (2.15) and (2.17) respectively, we must have, by (2.14), that \(8l - 8\) is less than the maximum difference between integers in the subdivided interval. This gives the required contradiction; we just give the details for \(a\) odd. In this case, the difference between integers in (2.17) in the subdivided interval \([(a - 1)^2, \ldots, a^2 - 1]\) is at most
\[
(a + 2c)(a - 2c + 2) - (a + 2c + 2)(a - 2c) = 8c < 4 + 8a^2 < 8p^{\frac{1}{2}} + 12 < 8p^{\frac{1}{2}} + 16 < 8l - 8.
\]

References


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