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# THE QUADRATIC AND QUARTIC CHARACTER OF CERTAIN QUADRATIC UNITS. II

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Let $m$ be a square-free integer greater than 1, and let $\epsilon_m$ denote the fundamental unit of the real quadratic field $Q(\sqrt{m})$. If $k$ is an integer not divisible by the odd prime $p$ and the Legendre symbol $(k/p)$ has the value 1, we define the symbol $(k/p)_4$ to be $+1$ or $-1$ according as $k$ is or is not a fourth power modulo $p$. Now if $(m/p) = +1$ we can interpret $\epsilon_m$ as an integer modulo $p$ and ask for the value of $(\epsilon_m/p)$. Because of the ambiguity in the choice of $\sqrt{m}$ taken modulo $p$ we must make sure that $(\epsilon_m/p)$ is well defined. This is the case if $\epsilon_m$ has norm $+1$ (written $N(\epsilon_m) = 1$) or if $N(\epsilon_m) = -1$ and $p \equiv 1 \pmod 4$. Whenever $(\epsilon_m/p) = 1$ we can ask for the value of $(\epsilon_m/p)_4$. This latter symbol is well defined if $N(\epsilon_m) = +1$ or if $N(\epsilon_m) = -1$ and $p \equiv 1 \pmod 8$. The evaluations of these symbols are generally given in terms of representations of a power of $p$ by certain positive-definite binary quadratic forms. This is convenient when considering applications to divisibility properties of recurrence sequences (see for example [**14**]).

An early result in this direction was proved by Barrucand and Cohn [1] who showed, using the arithmetic of $Q(\sqrt{-1}, \sqrt{2})$, that if $p \equiv 1 \pmod 8$ is prime, so that $p = c^2 + 8d^2$, then

$$(\epsilon_2/p) = (-1)^d.$$

This gives a criterion for the splitting of $p$ in the non-abelian number field $Q(\sqrt{-1}, \sqrt{2}, \sqrt{\epsilon_2})$.

Using similar methods, the present authors [17] have evaluated explicitly $(\epsilon_m/p)$ (when $N(\epsilon_m) = -1$) and $(\epsilon_m/p)_4$ (when $N(\epsilon_m) = +1$) for certain values of $m$, namely, those for which at least one of the imaginary bicyclic biquadratic fields

$$(1) \qquad Q(\sqrt{m}, \sqrt{-m}),\ Q(\sqrt{-m}, \sqrt{-2m}),\ \text{or}\ Q(\sqrt{-2m}, \sqrt{m}),$$

has class number one (21 fields in all, see [**6**]). In this paper we extend these results to an infinite class of values of $m$. Our results and conjectures arise from those of the above fields (1) which have class number not divisible by 4.

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TABLE (part 1)

NOTATION: $h = h(m, -m)$, $k =$ largest odd divisor of $h(m)$,
$l =$ largest odd divisor of $h(-m)$, * = conjectured

| CASE | $h$ | $h(m)$ | $h(-m)$ | $N(\epsilon_m)$ | $m$ | CHARACTERIZATION OF $m$ |
|---|---|---|---|---|---|---|
| 1.1 | $1 \pmod 2$ | $1 \pmod 2$ | $2 \pmod 4$ | $-1$ | $q$ | $q \equiv 5 \pmod 8$ |
| 1.2(i) | | | | | | |
| 1.2(ii) | $1 \pmod 2$ | $1 \pmod 2$ | $1 \pmod 2$ | $+1$ | $q$ | $q \equiv 3 \pmod 8$ |
| 1.2(iii) | | | | | | $q \equiv 7 \pmod 8$ |
| 1.3(i) | $2 \pmod 4$ | $2 \pmod 4$ | $2 \pmod 4$ | $-1$ | $2q$ | $q \equiv 5 \pmod 8$ |
| 1.3(ii) | | | | | | |
| 1.4(i) | $2 \pmod 4$ | $1 \pmod 2$ | $4 \pmod 8$ | $-1$ | $q$ | $q = c^2 + 8d^2 \equiv 1 \pmod 8$ |
| 1.4(ii) | | | | | | $(d \equiv 1 \pmod 2)$ |
| 1.5(i) | $2 \pmod 4$ | $1 \pmod 2$ | $2 \pmod 4$ | $+1$ | $2q$ | $q \equiv 3 \pmod 8$ |
| 1.5(ii) | | | | | | |
| 1.6(i)* | | | | | | $q \equiv 3 \pmod 8$, $q' \equiv 3 \pmod 4$ |
| 1.6(ii)* | $2 \pmod 4$ | $1 \pmod 2$ | $4 \pmod 8$ | $+1$ | $qq'$ | $\left(\frac{q'}{q}\right) = +1$ |
| 1.6(iii) | | | | | | |
| 1.7(i)* | | | | | | $q \equiv 5 \pmod 8$, $q' \equiv 3 \pmod 8$ |
| 1.7(ii)* | | | | | | $\left(\frac{q'}{q}\right) = -1$ |
| 1.7(iii)* | $2 \pmod 4$ | $2 \pmod 4$ | $2 \pmod 4$ | $+1$ | $qq'$ | |
| 1.7(iv)* | | | | | | $q \equiv 5 \pmod 8$, $q' \equiv 7 \pmod 8$ |
| 1.7(v)* | | | | | | $\left(\frac{q'}{q}\right) = -1$ |
| 1.7(vi)* | | | | | | |

| Congruential Character of Prime $p$ | Primitive Quadratic Partition of $p^{k\ell}$ | Character of Fundamental Unit |
|---|---|---|
| $\left(\frac{-1}{p}\right) = \left(\frac{q}{p}\right) = 1$ | $p^{k\ell} = x^2 + qy^2$ | $\left(\frac{\epsilon_q}{p}\right) = (-1)^y$ |
| $\left(\frac{-1}{p}\right) = \left(\frac{2}{p}\right) = \left(\frac{q}{p}\right) = 1$ | $p^{k\ell} = x^2 + 16qy^2$ | $\left(\frac{\epsilon_q}{p}\right)_4 = (-1)^y$ |
| | $4p^{k\ell} = x^2 + qy^2$ $(x \equiv 1 \pmod 4)$ | $\left(\frac{\epsilon_q}{p}\right)_4 = (-1)^{\frac{x-1}{4}+\frac{p-1}{8}}$ |
| | $p^{k\ell} = x^2 + 16qy^2$ | $\left(\frac{\epsilon_q}{p}\right)_4 = (-1)^y$ |
| $\left(\frac{-1}{p}\right) = \left(\frac{2}{p}\right) = \left(\frac{q}{p}\right) = 1$ | $p^{k\ell} = x^2 + 8qy^2$ | $\left(\frac{\epsilon_{2q}}{p}\right) = (-1)^y$ |
| $\left(\frac{-1}{p}\right) = 1, \left(\frac{2}{p}\right) = \left(\frac{q}{p}\right) = -1$ | $p^{k\ell} = 8x^2 + qy^2$ | $\left(\frac{\epsilon_{2q}}{p}\right) = (-1)^{x+1}$ |
| $\left(\frac{-1}{p}\right) = \left(\frac{q}{p}\right) = 1$ | $p^{k\ell} = x^2 + qy^2$ | $\left(\frac{\epsilon_q}{p}\right) = +1$ |
| | $2p^{k\ell} = x^2 + qy^2$ | $\left(\frac{\epsilon_q}{p}\right) = -1$ |
| $\left(\frac{-1}{p}\right) = \left(\frac{2}{p}\right) = \left(\frac{q}{p}\right) = 1$ | $p^{k\ell} = x^2 + 8qy^2$ | $\left(\frac{\epsilon_{2q}}{p}\right)_4 = (-1)^y$ |
| $\left(\frac{-1}{p}\right) = 1, \left(\frac{2}{p}\right) = \left(\frac{q}{p}\right) = -1$ | | $\left(\frac{\epsilon_{2q}}{p}\right) = -1$ |
| $\left(\frac{-1}{p}\right) = \left(\frac{q}{p}\right) = \left(\frac{q'}{p}\right) = 1$ | $p^{k\ell} = x^2 + 4qq'y^2$ | $\left(\frac{\epsilon_{qq'}}{p}\right)_4 = (-1)^y$ |
| | $p^{k\ell} = 4x^2 + qq'y^2$ | $\left(\frac{\epsilon_{qq'}}{p}\right)_4 = \left(\frac{-2}{q'}\right)(-1)^{x+1}$ |
| $\left(\frac{-1}{p}\right) = 1, \left(\frac{q}{p}\right) = \left(\frac{q'}{p}\right) = -1$ | | $\left(\frac{\epsilon_{qq'}}{p}\right) = -1$ |
| $\left(\frac{-1}{p}\right) = \left(\frac{2}{p}\right) = \left(\frac{q}{p}\right) = \left(\frac{q'}{p}\right) = 1$ | $p^{k\ell} = x^2 + 16qq'y^2$ | $\left(\frac{\epsilon_{qq'}}{p}\right)_4 = (-1)^y$ |
| $\left(\frac{-1}{p}\right) = 1, \left(\frac{2}{p}\right) = \left(\frac{q}{p}\right) = \left(\frac{q'}{p}\right) = -1$ | $p^{k\ell} = qx^2 + 16q'y^2$ | $\left(\frac{\epsilon_{qq'}}{p}\right)_4 = (-1)^{y+1}$ |
| $\left(\frac{-1}{p}\right) = \left(\frac{2}{p}\right) = \left(\frac{q}{p}\right) = \left(\frac{q'}{p}\right) = 1$ | $p^{k\ell} = x^2 + 16qq'y^2$ | $\left(\frac{\epsilon_{qq'}}{p}\right)_4 = (-1)^y$ |
| | $4p^{k\ell} = x^2 + qq'y^2$ $(x \equiv 1 \pmod 4)$ | $\left(\frac{\epsilon_{qq'}}{p}\right)_4 = (-1)^{\frac{x-1}{4}+\frac{p-1}{8}}$ |
| $\left(\frac{-1}{p}\right) = 1, \left(\frac{2}{p}\right) = \left(\frac{q}{p}\right) = \left(\frac{q'}{p}\right) = -1$ | $p^{k\ell} = qx^2 + 16q'y^2$ | $\left(\frac{\epsilon_{qq'}}{p}\right)_4 = (-1)^y$ |
| | $4p^{k\ell} = qx^2 + q'y^2$ $(y \equiv 1 \pmod 4)$ | $\left(\frac{\epsilon_{qq'}}{p}\right)_4 = (-1)^{\frac{y-1}{4}+\frac{p+3}{8}}$ |

TABLE (part 2)

NOTATION: $h = h(-m, -2m)$, $k =$ largest odd divisor of $h(-m)$,
$\ell =$ largest odd divisor of $h(-2m)$, * = conjectured
NOTE: $m$ and $2m$ give rise to the same field.

| CASE | $h$ | $h(-m)$ | $h(-2m)$ | $N(\epsilon_m)$ | $m$ | CHARACTERIZATION OF $m$ |
|---|---|---|---|---|---|---|
| 2.1(i) | 1(mod 2) | 1(mod 2) | 2(mod 4) | $-1$ | $q$ | $q \equiv 3 \pmod 8$ |
| 2.1(ii) | | | | | | |
| 2.2 | 2(mod 4) | 2(mod 4) | 2(mod 4) | $-1$ | $q$ | $q \equiv 5 \pmod 8$ |
| 2.3(i)* | 2(mod 4) | 1(mod 2) | 4(mod 8) | $-1$ | $q$ | $q \equiv 7 \pmod{16}$ |
| 2.3(ii)* | | | | | | |

TABLE (part 3)

NOTATION: $h = h(-2m, m)$, $k =$ largest odd divisor of $h(m)$,
$\ell =$ largest odd divisor of $h(-2m)$, * = conjectured

| CASE | $h$ | $h(-2m)$ | $h(m)$ | $N(\epsilon_m)$ | $m$ | CHARACTERIZATION OF $m$ |
|---|---|---|---|---|---|---|
| 3.1 | 1(mod 2) | 2(mod 4) | 1(mod 2) | $-1$ | $q$ | $q \equiv 5 \pmod 8$ |
| 3.2(i) | 1(mod 2) | 1(mod 2) | 1(mod 2) | $+1$ | $2q$ | $q \equiv 3 \pmod 8$ |
| 3.2(ii) | | | | | | |
| 3.2(iii) | | | | | | $q \equiv 7 \pmod 8$ |
| 3.3 | 2(mod 4) | 2(mod 4) | 2(mod 4) | $-1$ | $2q$ | $q \equiv 5 \pmod 8$ |
| 3.4(i) | 2(mod 4) | 4(mod 8) | 1(mod 2) | $-1$ | $q$ | $q = r^2 - 8s^2 \equiv 1 \pmod 8$ |
| 3.4(ii) | | | | | | $(r > 0,\ r \equiv 5, 7 \pmod 8)$ |
| 3.5 | 2(mod 4) | 2(mod 4) | 1(mod 2) | $+1$ | $q$ | $q \equiv 3 \pmod 8$ |
| 3.6(i)* | 2(mod 4) | 4(mod 8) | 1(mod 2) | $+1$ | $qq'$ | $q \equiv 3 \pmod 8$, $q' \equiv 7 \pmod 8$ $\left(\frac{q'}{q}\right) = +1$ |
| 3.6(ii) | | | | | | |
| 3.7(i)* | 2(mod 4) | 2(mod 4) | 2(mod 4) | $+1$ | $2qq'$ | $q \equiv 5 \pmod 8$, $q' \equiv 3 \pmod 8$ $\left(\frac{q'}{q}\right) = -1$ |
| 3.7(ii)* | | | | | | $q \equiv 5 \pmod 8$ $q' \equiv 7 \pmod 8$ $\left(\frac{q'}{q}\right) = -1$ |
| 3.7(iii)* | | | | | | |

| Congruental Character of Prime $p$ | Primitive Quadratic Partition of $p^{kf}$ | Character of Fundamental Unit |
|---|---|---|
| $\left(\frac{-1}{p}\right)=\left(\frac{2}{p}\right)=\left(\frac{q}{p}\right)=1$ | $p^{kf}=x^2+8qy^2$<br>$p^{kf}=a^2+16qb^2$ | $\left(\frac{\epsilon_2}{p}\right)=(-1)^{y+b+\frac{p-1}{8}}$ |
| | $p^{kf}=x^2+8qy^2$<br>$4p^{kf}=a^2+qb^2$<br>$(a\equiv 1 \pmod 4)$ | $\left(\frac{\epsilon_2}{p}\right)=(-1)^{y+\frac{a-1}{4}}$ |
| $\left(\frac{-1}{p}\right)=\left(\frac{2}{p}\right)=\left(\frac{q}{p}\right)=1$ | $p^{kf}=x^2+8qy^2$<br>$=a^2+qb^2$ | $\left(\frac{\epsilon_2}{p}\right)=(-1)^{y+b}$ |
| $\left(\frac{-1}{p}\right)=\left(\frac{2}{p}\right)=\left(\frac{q}{p}\right)=1$ | $p^{kf}=x^2+8qy^2$<br>$=a^2+16qb^2$ | $\left(\frac{\epsilon_2}{p}\right)=\left(\frac{2}{p}\right)_4(-1)^b$ |
| | $p^{kf}=2x^2+qy^2$<br>$=a^2+16qb^2$ | $\left(\frac{\epsilon_2}{p}\right)=\left(\frac{2}{p}\right)_4(-1)^{b+1}$ |

| Congruental Character of Prime $p$ | Primitive Quadratic Partition of $p^{kf}$ | Character of Fundamental Unit |
|---|---|---|
| $\left(\frac{-1}{p}\right)=\left(\frac{2}{p}\right)=\left(\frac{q}{p}\right)=1$ | $p^{kf}=x^2+8qy^2$<br>$=c^2+8d^2$ | $\left(\frac{\epsilon_q}{p}\right)=(-1)^{y+d}$ |
| $\left(\frac{-1}{p}\right)=\left(\frac{2}{p}\right)=\left(\frac{q}{p}\right)=1$ | $p^{kf}=x^2+16qy^2$<br>$=c^2+8d^2$ | $\left(\frac{\epsilon_{2q}}{p}\right)_4=(-1)^{y+d+\frac{p-1}{8}}$ |
| | $4p^{kf}=x^2+qy^2$<br>$(x\equiv 1 \pmod 4)$<br>$p^{kf}=c^2+8d^2$ | $\left(\frac{\epsilon_{2q}}{p}\right)_4=(-1)^{\frac{x-1}{4}+d}$ |
| | $p^{kf}=x^2+16qy^2$ | $\left(\frac{\epsilon_{2q}}{p}\right)_4=(-1)^{y+\frac{p-1}{8}}$ |
| $\left(\frac{-1}{p}\right)=\left(\frac{2}{p}\right)=\left(\frac{q}{p}\right)=1$ | $p^{kf}=x^2+qy^2$<br>$=c^2+8d^2$ | $\left(\frac{\epsilon_{2q}}{p}\right)=(-1)^{y+d}$ |
| $\left(\frac{-1}{p}\right)=\left(\frac{2}{p}\right)=\left(\frac{q}{p}\right)=1$ | $p^{kf}=x^2+8qy^2$<br>$=c^2+8d^2$ | $\left(\frac{\epsilon_q}{p}\right)=(-1)^{y+d}$ |
| | $p^{kf}=2x^2+qy^2$<br>$=c^2+8d^2$ | $\left(\frac{\epsilon_q}{p}\right)=(-1)^{x+d+D}$<br>(where $q=C^2+8D^2$) |
| $\left(\frac{-1}{p}\right)=\left(\frac{2}{p}\right)=\left(\frac{q}{p}\right)=1$ | $p^{kf}=x^2+8qy^2$<br>$=c^2+8d^2$ | $\left(\frac{\epsilon_q}{p}\right)_4=(-1)^{y+d+\frac{p-1}{8}}$ |
| $\left(\frac{-1}{p}\right)=\left(\frac{2}{p}\right)=\left(\frac{q}{p}\right)=\left(\frac{q'}{p}\right)=1$ | $p^{kf}=x^2+8qq'y^2$<br>$=c^2+8d^2$ | $\left(\frac{\epsilon_{qq'}}{p}\right)_4=\left(\frac{\epsilon_{2q'}}{p}\right)_4(-1)^{y+d}$ |
| $\left(\frac{-1}{p}\right)=1,\left(\frac{q}{p}\right)=\left(\frac{q'}{p}\right)=-1$ | | $\left(\frac{\epsilon_{qq'}}{p}\right)=-1$ |
| $\left(\frac{-1}{p}\right)=\left(\frac{2}{p}\right)=\left(\frac{q}{p}\right)=\left(\frac{q'}{p}\right)=1$ | $p^{kf}=x^2+16qq'y^2$ | $\left(\frac{\epsilon_{2qq'}}{p}\right)_4=\left(\frac{\epsilon_{2q}}{p}\right)\left(\frac{\epsilon_{q'}}{p}\right)_4(-1)^y$ |
| | $p^{kf}=x^2+16qq'y^2$ | $\left(\frac{\epsilon_{2qq'}}{p}\right)_4=\left(\frac{\epsilon_{2q'}}{p}\right)_4\left(\frac{2}{p}\right)_4(-1)^y$ |
| | $4p^{kf}=x^2+qq'y^2$<br>$(x\equiv 1 \pmod 4)$ | $\left(\frac{\epsilon_{2qq'}}{p}\right)_4=\left(\frac{\epsilon_{2q'}}{p}\right)_4\left(\frac{2}{p}\right)_4(-1)^{\frac{x-1}{4}+\frac{p-1}{8}}$ |

We begin by characterizing such fields (excluding the field $Q(\sqrt{-1}, \sqrt{2})$). Our starting point is a formula of Hergoltz [13]; if $K \neq Q(\sqrt{-1}, \sqrt{2})$ is an imaginary bicyclic biquadratic field with class number $H$, $k_1$, $k_2$, $k_3$ its three quadratic subfields with $k_3$ real ($k_i$ having class number $h_i$), then

$$H = \frac{h_1 h_2 h_3}{\lambda_0},$$

where for $\theta$ and $\epsilon$ fundamental units of $K$ and $k_3$ respectively we have $N_{K/k_3}(\theta) = \epsilon^{\lambda_0}$. Then, using various divisibility results on class numbers of quadratic fields [1], [3], [4], [5], [7], [8], [10], [11], [12], together with certain elementary properties of the fundamental unit $\epsilon_m$ [8], [18], we obtain (after some calculation) the first six columns of the table, where $h(m)$ denotes the class number of $Q(\sqrt{m})$ and $h(m, n)$ the class number of $Q(\sqrt{m}, \sqrt{n})$.

We have been able to obtain results on either the quadratic character or the quartic character of $\epsilon_m$ in all of the cases listed in the table, except those marked with an asterisk, where we have only conjectures (see final three columns of table). We emphasize that all quadratic partitions indicated in the table are primitive ones, that is, the values of the variables are coprime, and the $q$ and $q'$ denote odd distinct primes.

We illustrate the ideas involved by treating case 1.3 (ii). In this case $q \equiv 5 \pmod 8$ is prime and $p$ is a prime satisfying $(-1/p) = 1$, $(2/p) = (q/p) = -1$. Then in $Q(\sqrt{2q}, \sqrt{-2q})$ we have the prime ideal factorizations $(p) = P P' \bar{P} \bar{P}'$ and $(2) = Q^4$, where $P$, $P'$, $\bar{P}$, $\bar{P}'$ are distinct conjugate prime ideals and $Q$ is a prime ideal (see for example [21]). Here $'$ denotes conjugation with respect to $\sqrt{2q}$ and $\bar{\ }$ with respect to $\sqrt{-1}$. Since $Q(\sqrt{2q}, \sqrt{-2q})$ has class number $2kl$, where $k = (1/2)h(2q)$ and $l = (1/2)h(-2q)$ are odd, there is a unique ideal class $C$ such that $C$ has order 2 in the ideal class group. Moreover $Q \in C$ as $Q^2$ is principal while $Q$ is non-principal. As $P^{2kl}$ is principal, either $P^{kl}$ is principal or $P^{kl}$ is equivalent to $Q$ and $QP^{kl}$ is principal. Suppose $P^{kl} = (\alpha)$. Then taking norms we have $(p^{kl}) = (\alpha\alpha'\bar{\alpha}\bar{\alpha}')$, which leads to $p^{kl} = x^2 + 2qy^2$, which is impossible as $q \equiv 5 \pmod 8$. Therefore we have $QP^{kl} = (\alpha)$, where $\alpha$ is an integer of $Q(\sqrt{2q}, \sqrt{-2q})$, so that [23]

$$\alpha = A + \frac{B}{2}\sqrt{2q} + C\sqrt{-1} + \frac{D}{2}\sqrt{-2q},$$

where $A$, $B$, $C$, $D$ are rational integers with $B \equiv D \pmod 2$. Then we have

$$\alpha\bar{\alpha} = \{(A^2 + C^2) + \frac{q}{2}(B^2 + D^2)] + (AB + CD)\sqrt{2q}$$

and

$$\alpha\bar{\alpha}' = \{(A^2 + C^2) - \frac{q}{2}(B^2 + D^2)\} + (AD - BC)\sqrt{-2q}$$

so that

$$p^{k\ell} = 8x^2 + qy^2 = 2u^2 - qv^2$$

with

$$x = \frac{1}{4}(A^2 + C^2 - \frac{q}{2}(B^2 + D^2)),\ \ y = AD - BC,$$

$$u = \frac{1}{2}(A^2 + C^2 + \frac{q}{2}(B^2 + D^2)) > 0,\ \ v = AB + CD.$$

Note that the possibility $-p^{k\ell} = 2u^2 - qv^2$ cannot occur as $\alpha\alpha'\bar{\alpha}\bar{\alpha}' > 0$. Moreover $(x, y) = (u, v) = 1$, for if $(x, y) > 1$ ($(u, v) > 1$ can be treated similarly) we have $p|x$, $p|y$, so that $p|4x + y\sqrt{-2q} = \alpha\bar{\alpha}'$ giving $(p) \mid (\alpha)(\bar{\alpha}')$, $PP'\bar{P}\bar{P}' \mid Q^2P^{k\ell}\bar{P}^{k\ell}$, that is $P' \mid P$ or $\bar{P}'$, since $P'$ is a prime ideal coprime with $Q$, contradicting that $P$, $P'$, $\bar{P}$, $\bar{P}'$ are distinct.

Next from $p^{k\ell} = 2u^2 - qv^2$ $(u > 0)$ we have $\sqrt{2q} \equiv 2u/v \pmod p$ so that

$$\left(\frac{\epsilon_{2q}}{p}\right) = \left(\frac{T + U\sqrt{2q}}{p}\right) = \left(\frac{T + U\,2u/v}{p}\right)$$
$$= \left(\frac{v}{p}\right)\left(\frac{Tv + 2Uu}{p}\right).$$

Choosing, without loss of generality, $v > 0$, we have $(v/p) = (p/v) = (p^{k\ell}/v) = (2u^2/v) = (2/v)$, as $p \equiv 1 \pmod 4$ and $k\ell$ odd.

Next we have

$$\left(\frac{Tv + 2Uu}{p}\right) = \left(\frac{p}{Tv + 2Uu}\right) = \left(\frac{p^{k\ell}}{Tv + 2Uu}\right)$$
$$= \left(\frac{2u^2 - qv^2}{Tv + 2Uu}\right)$$
$$= \left(\frac{-2}{Tv + 2Uu}\right)\left(\frac{2qv^2 - 4u^2}{Tv + 2Uu}\right)$$
$$= \left(\frac{-2}{Tv + 2Uu}\right),$$

as $(2qv^2 - 4u^2)/r = 1$ for each prime factor $r$ of $Tv + 2Uu$.

Hence we have

$$\left(\frac{\epsilon_{2q}}{p}\right) = \left(\frac{2}{v}\right)\left(\frac{-2}{Tv + 2Uu}\right).$$

Now as $T \equiv \pm 3 \pmod 8$, $U \equiv 1 \pmod 4$, we have

$$\left(\frac{\epsilon_{2q}}{p}\right) = \left(\frac{2}{v}\right)\left(\frac{-2}{3v + 2u}\right) = \left(\frac{2}{v}\right)\left(\frac{-2}{v + 6u}\right).$$

Consideration of cases gives

$$\left(\frac{\epsilon_{2q}}{p}\right) = -\left(\frac{-1}{u}\right).$$

Next as $A$ and $C$ are of opposite parity (since $v$ is odd) we have

$$u + 2x = A^2 + C^2 \equiv 1 \pmod 4)$$

and so

$$\left(\frac{-1}{u}\right) = (-1)^x$$

giving

$$\left(\frac{\epsilon_{2q}}{p}\right) = (-1)^{x+1}.$$

Some special cases of our results are due originally to Brandler [2] and Lehmer [15], [16]. For example case 1.4 was proved by Brandler for $q = 17$ and by Lehmer [15] for $q = 17$, 73, 97 and 193, see also Parry [19], while case 3.4 can be thought of as giving an explicit form of some results ([15], Theorems 2 and 3) of Lehmer.

We note that by combining cases 1.1, 1.3 and 2.2 we obtain, for primes $q \equiv 5 \pmod 8$ the relation $(\epsilon_{2q}/p) = (\epsilon_2/p)(\epsilon_q/p)$, which is a special case of a remark of Barrucand (noted in [15]) on a result of Rédei ([20], equation (30)), as well as a special case of a theorem of Furuta [9]. (See also a related paper of Williams [22]). Combining cases 1.2 and 3.2 we obtain the following analogous result relating the biquadratic characters of $\epsilon_q$ and $\epsilon_{2q}$ in case $q \equiv 3 \pmod 4$.

THEOREM 1. *Suppose $q$ is a prime with $q \equiv 3 \pmod 4$. Let $p$ be a prime such that $(-1/p) = (2/p) = (q/p) = 1$. Then we have*

$$\left(\frac{\epsilon_{2q}}{p}\right)_4 = \begin{cases} \left(\frac{2}{p}\right)_4 \left(\frac{\epsilon_q}{p}\right)_4, & \text{if } q \equiv 3 \pmod 8, \\ \left(\frac{-1}{p}\right)_8 \left(\frac{\epsilon_q}{p}\right)_4, & \text{if } q \equiv 7 \pmod 8. \end{cases}$$

For primes $q \equiv 5 \pmod 8$, we may use the relationship $(\epsilon_{2q}/p) = (\epsilon_2/p)(\epsilon_q/p)$ in conjunction with cases 1.1, 2.2, and 3.3 to establish another relationship, a direct proof of which would seem to require the arithmetic of the octic extension $Q(\sqrt{q}, \sqrt{-1}, \sqrt{2})$.

THEOREM 2. *Suppose $q$ is a prime, $q \equiv 5 \pmod 8$. Let $p$ be a prime with $(-1/p) = (2/p) = (q/p) = 1$ so that*

$$p^{k\ell} = x^2 + qy^2 = a^2 + 8qb^2 = c^2 + 8d^2,$$

*with $(x, y) = (a, b) = (c, d) = 1$. Then we have*

$$y + b + d \equiv 0 \pmod 2.$$

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