We present here a generalization of Apollonius' theorem which makes an interesting geometric application of vector algebra. *Let ABCD be any quadrilateral. If M and N are the midpoints of the diagonals AC and BD, respectively, then* $(AB)^2+(BC)^2+(CD)^2+(DA)^2=(AC)^2+(BD)^2+4(MN)^2$ (*see* FIGURE 1). Basically $4(MN)^2$ is the appropriate correction factor in the case that the two diagonals do not bisect each other. To see this we use a coordinate system with A at the origin and think of each of the points B, C, D, M, N as a vector $\beta, \gamma, \delta, \mu, \nu$ respectively. Now $\mu=\frac{1}{2}\gamma$ and $\nu=\frac{1}{2}(\beta+\delta)$, so by direct computation

$$4(MN)^2=\|\beta+\delta-\gamma\|^2=\|\beta\|^2+\|\delta\|^2+\|\beta-\gamma\|^2+\|\delta-\gamma\|^2-\|\beta-\delta\|^2-\|\gamma\|^2$$

which is our original statement in vector language.

Reference

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The Quadratic Character of 2 mod p

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Here is a very simple proof of the result that

$$\left(\frac{2}{p}\right)=(-1)^{(p^2-1)/8}$$

for p an odd prime, where the Legendre symbol $(2|p)$ is $+1$ if 2 is a perfect square mod p and -1 otherwise. In other words, 2 is a perfect square mod p if and only if $(p^2-1)/8$ is even, i.e., if and only if $p\equiv 1$ or 7 (mod 8). The idea is to look at the number N_p of ordered pairs (x, y) — incongruent modulo p — which satisfy

(1) $$x^2+y^2\equiv 4 \pmod p,\quad x\not\equiv 0 \pmod p \quad\text{and}\quad y\not\equiv 0 \pmod p.$$

As the number of z with $z^2\equiv 2 \pmod p$ is given by $1+(2|p)$, the number of solutions (x, y) of (1) with $x\equiv \pm y \pmod p$ is $2(1+(2|p))$. Now each solution (x, y) with $x\not\equiv y \pmod p$ (if any) gives rise to eight distinct solutions of (1), namely $(\pm x, \pm y)$, $(\pm y, \pm x)$, so that we have

(2) $$N_p\equiv 2+2\left(\frac{2}{p}\right) \pmod 8.$$

Next, transforming the variables x, y to y, t by means of the transformation $x\equiv(2-y)t \pmod p$ we see that all the solutions of (1) are given by

$$(x, y)\equiv\left(\frac{4t}{t^2+1}, \frac{2(t^2-1)}{t^2+1}\right) \pmod p$$

with $2\leqq t\leqq p-2$, $t^2\not\equiv -1 \pmod p$. Thus we have

(3) $$N_p=p-3-\left\{1+\left(\frac{-1}{p}\right)\right\}=p-4-(-1)^{(p-1)/2}.$$

Putting (2) and (3) together we obtain

$$\left(\frac{2}{p}\right) \equiv \tfrac{1}{2}(p-(-1)^{(p-1)/2})-3 \pmod 4$$

$$\equiv \begin{cases} +1 \pmod 4, \text{ if } p \equiv 1,7 \pmod 8, \\ -1 \pmod 4, \text{ if } p \equiv 3,5 \pmod 8. \end{cases}$$

As $\left(\frac{2}{p}\right) = \pm 1$ and $\frac{p^2-1}{8} \equiv \begin{cases} 0 \pmod 2 \Leftrightarrow p \equiv 1,7 \pmod 8, \\ 1 \pmod 2 \Leftrightarrow p \equiv 3,5 \pmod 8, \end{cases}$

the required result follows.

Spaces in which Compact Sets are Closed

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It is known that if the graph $G(g)$ of a function $g\colon X \to Y$ is compact and compact subsets of X are closed (compact subsets of Y are closed), then g is continuous (closed) ([**1**], [**2**]). In this note, we prove the following:

THEOREM. *If X is a compact space, the following statements are equivalent:*
(1) *Compact subsets of X are closed.*
(2) *Any function with a compact graph from X to a space is continuous.*
(3) *Any function with a compact graph from a space to X is closed.*

Proof. In what follows, let π_x and π_y be the projections from $X \times Y$ onto X and Y respectively. To show that (1) implies (2), let $g\colon X \to Y$ be a function with a compact graph and let $A \subset Y$ be closed; since π_y is continuous, $\pi_y^{-1}(A) \cap G(g)$ is compact and so the image of this set under π_x is compact; so $g^{-1}(A) = \pi_x[\pi_y^{-1}(A) \cap G(g)]$ is compact in X and thus closed in X. To see that (2) implies (3) let $g\colon Y \to X$ have a compact graph, $G(g)$. Let $A \subset Y$ be closed. Then $\pi_y^{-1}(A) \cap G(g)$ is compact, so $g(A) = \pi_x(\pi_y^{-1}(A) \cap G(g))$ is compact in X. If T is the topology on X, then X is compact with the simple extension, $T(g(A))$, of T through the compact set $g(A)$ and $g(A)$ is $T(g(A))$-closed (see [**3**]). The identity function i from (X, T) to $(X, T(g(A)))$ has a compact graph since $T \subset T(g(A))$ and $T(g(A)) \subset T(g(A))$ renders the function h from $(X, T(g(A)))$ to $X \times X$ defined by $h(x) = (x, x)$ continuous and since $h(X) = G(i)$. Thus i is continuous from (2) so $g(A) = i^{-1}(g(A))$ is T-closed in X. Finally, to verify that (3) implies (1), let $A \subset X$ be compact.

$G(i)$ is compact for the identity function i from $(X, T(A))$ to X (same reasoning as above) so i is closed. Since A is $T(A)$-closed, A is closed in X. This completes the proof.

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