NOTE ON THE KLOOSTERMAN SUM

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ABSTRACT. The Kloosterman sum

\[ \sum_{x=0}^{p-1} \exp(2\pi i n x / p^\alpha), \]

where \( p \) is an odd prime, \( \alpha \geq 2 \) and \( (n, p) = 1 \), is evaluated in a very short direct way.

Let \( p \) denote an odd prime, \( \alpha \) a positive integer, and \( n \) an integer such that \( (n, p) = 1 \). The Kloosterman sum \( A_{\nu \alpha}(n) \) is given by

\[ A_{\nu \alpha}(n) = \sum_{\bar{x}=0}^{p^\alpha-1} \exp(2\pi i n (x + \bar{x}) / p^\alpha), \]

where the dash (') indicates that \( x \) only takes values from 0, 1, \( \cdots \), \( p^\alpha-1 \) which are coprime with \( p \), and \( \bar{x} \) is the unique solution of the congruence \( x\bar{x} \equiv 1 \pmod{p^\alpha} \) satisfying \( 0 < \bar{x} < p^\alpha \). Salié [3] has evaluated \( A_{\nu \alpha}(n) \) explicitly when \( \alpha \geq 2 \). His evaluation is based upon induction. A direct (but fairly long) proof has been given by Whiteman [4] which requires results concerning Ramanujan sums. In this note we give a modification of Salié’s original argument which gives a very short direct evaluation of \( A_{\nu \alpha}(n) \). (The referee has kindly pointed out that essentially the same technique has been used by Estermann [2], Carlitz [1].)

We let \( \gamma = \alpha - [\alpha/2] \) and \( \delta = [\alpha/2] \), so that \( \alpha = \gamma + \delta \), \( 2\gamma \geq \alpha \) and \( \gamma \geq \delta \geq 1 \). Setting \( x = u + v p^\gamma \) \( (u = 0, 1, \cdots, p^\gamma - 1; v = 0, 1, \cdots, p^\delta - 1) \) in (1), so that \( \bar{x} = \bar{u} - \bar{u}^2 v p^\gamma \pmod{p^\alpha} \), we obtain

\[ A_{\nu \alpha}(n) = \sum_{u=0}^{p^\gamma-1} \sum_{v=0}^{p^\delta-1} \exp(2\pi i n (u + v p^\gamma + (\bar{u} - \bar{u}^2 v p^\gamma)) / p^\alpha) \]

\[ = \sum_{u=0}^{p^\gamma-1} \exp(2\pi i n u / p^\alpha) \sum_{v=0}^{p^\delta-1} \exp(2\pi i n v (1 - \bar{u}^2) / p^\delta) \]

\[ = p^\delta \sum_{u=0; u^2 \equiv 1 \pmod{p^\delta}}^{p^\gamma-1} \exp(2\pi i n (u + \bar{u}) / p^\alpha). \]

If \( \alpha \) is even, say \( \alpha = 2\beta \), then \( \gamma = \beta = \delta \), and as the solutions \( u \) of \( u^2 \equiv 1 \pmod{p^\delta} \) in the range \( 0 \leq u \leq p^\delta - 1 \) are \( u = 1, p^\delta - 1 \) (so that
\[ \bar{a} = 1, \ p^{2\beta} - p^\delta - 1 \text{ respectively} \] we have

\[ A_p^{\alpha}(n) = p^\delta \left\{ \exp\left(4\pi in/p^{2\beta}\right) + \exp\left(-4\pi in/p^{2\beta}\right) \right\} = 2p^\delta \cos\left(4\pi n/p^{2\beta}\right). \]

If \( \alpha \) is odd, say \( \alpha = 2\beta + 1 \), then \( \gamma = \beta + 1, \ \delta = \beta, \) and as the solutions \( u \) of \( u^2 \equiv 1 \ (\mod p^\beta) \) in the range \( 0 \leq u \leq p^{\beta+1} - 1 \) are \( u = 1 + wp^\beta \) \( (w = 0, 1, \ldots, p-1), \ -1 + wp^\beta \ (w = 1, 2, \ldots, p) \) (so that \( \bar{a} = 1 - wp^\beta + wp^\beta, \ -1 - wp^\beta - wp^\beta \ (\mod p^{2\beta+1}) \) respectively) we have

\[ A_{p^{2\beta+1}}(n) = p^\beta \left\{ \exp\left(4\pi in/p^{2\beta+1}\right) \sum_{w=0}^{p-1} \exp\left(2\pi i w^2/p\right) + \exp\left(-4\pi in/p^{2\beta+1}\right) \sum_{w=1}^{p} \exp\left(-2\pi i w^2/p\right) \right\} \]

\[ = 2(n/p) p^{\beta+1/2} \cos\left(4\pi n/p^{2\beta+1}\right), \quad \text{if } p \equiv 1 \ (\mod 4), \]

\[ = -2(n/p) p^{\beta+1/2} \sin\left(4\pi n/p^{2\beta+1}\right), \quad \text{if } p \equiv 3 \ (\mod 4), \]

using the well-known result [4]

\[ \sum_{w=0}^{p-1} \exp\left(2\pi i w^2/p\right) = (n/p) i^{(p-1)/4} p^{1/2}. \]

REFERENCES


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