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On Salié's Sum

KENNETH S. WILLIAMS*

Department of Mathematics, Carleton University, Ottawa 1, Canada

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A finite transformation formula involving the Legendre symbol is proved, from which the value of Salié's sum $\sum_{x=1}^{p-1} (x/p)\, e^{2\pi i k(x+\bar{x})/p}$ can be deduced immediately.

Let p denote an odd prime. If $F(x)$ is a function with period p it is well known that

$$\sum_{x=1}^{p-1} F(x) + \sum_{x=1}^{p-1} \left(\frac{x}{p}\right) F(x) = \sum_{x=1}^{p-1} F(x^2), \tag{1}$$

and, if $a \not\equiv 0 \pmod p$, Jacobsthal [1] has noted that

$$\sum_{x=0}^{p-1} F(x) + \sum_{x=0}^{p-1} \left(\frac{x^2-4a}{p}\right) F(x) = \sum_{x=1}^{p-1} F(x + a\bar{x}), \tag{2}$$

where $\left(\frac{x}{p}\right)$ is the Legendre symbol and $\bar{x}$ is the unique integer such that $x\bar{x} \equiv 1$ (all congruences in this note are to be taken mod p) and $0 < \bar{x} < p$. It is the purpose of this note to prove a result of a similar type to (1) and (2), namely,

$$\sum_{x=0}^{p-1} \left(\frac{x-2}{p}\right) F(x) + \sum_{x=0}^{p-1} \left(\frac{x+2}{p}\right) F(x) = \sum_{x=0}^{p-1} \left(\frac{x}{p}\right) F(x+\bar{x}). \tag{3}$$

The value of the Salié sum [3] can be deduced from (3) since with the obvious choice of $F(x)$ we obtain the Salié sum as the sum of two Gaussian sums.

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Taking $a = 1$ in (2) and replacing $F(x)$ by $\left(\frac{x-2}{p}\right) F(x)$ we obtain

$$\sum_{x=0}^{p-1} \left(\frac{x-2}{p}\right) F(x) + \sum_{x=0}^{p-1} \left(\frac{x+2}{p}\right)\left(\frac{(x-2)^2}{p}\right) F(x)$$

$$= \sum_{x=1}^{p-1} \left(\frac{x-2+\bar{x}}{p}\right) F(x+\bar{x}),$$

i.e.,

$$\sum_{x=0}^{p-1} \left(\frac{x-2}{p}\right) F(x) + \sum_{\substack{x=0 \\ x \neq 2}}^{p-1} \left(\frac{x+2}{p}\right) F(x)$$

$$= \sum_{x=1}^{p-1} \left(\frac{x}{p}\right)\left(\frac{(x-1)^2}{p}\right) F(x+\bar{x}) = \sum_{x=2}^{p-1} \left(\frac{x}{p}\right) F(x+\bar{x}). \qquad (4)$$

This proves (3) as $\left(\frac{4}{p}\right) F(2) = \left(\frac{1}{p}\right) F(1+\bar{1})$.

Taking $F(x) = e(kx)$ $(k \not\equiv 0)$ in (3), where $e(t) = e^{2\pi i t/p}$, we obtain

$$\sum_{x=1}^{p-1} \left(\frac{x}{p}\right) e(k(x+\bar{x})) = \sum_{x=0}^{p-1} \left(\frac{x-2}{p}\right) e(kx) + \sum_{x=0}^{p-1} \left(\frac{x+2}{p}\right) e(kx),$$

which gives Salié's result [3]

$$\sum_{x=1}^{p-1} \left(\frac{x}{p}\right) e(k(x+\bar{x})) = \left(\frac{k}{p}\right) i^{\frac{1}{4}(p-1)^2} p^{\frac{1}{2}} (e(2k) + e(-2k)),$$

in view of the Gaussian sum [2]

$$\sum_{x=0}^{p-1} \left(\frac{x-a}{p}\right) e(kx) = \left(\frac{k}{p}\right) i^{\frac{1}{4}(p-1)^2} p^{\frac{1}{2}}\, e(ka).$$

References

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3. H. Salié, Über die Kloostermanschen Summen $S(u, v; q)$, *Math. Z.* **34** (1931), 91–109.