On Salié's Sum

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A finite transformation formula involving the Legendre symbol is proved, from which the value of Salié's sum \( \sum_{x=1}^{p-1} (x/p) e^{2\pi i (x+\bar{x})/p} \) can be deduced immediately.

Let \( p \) denote an odd prime. If \( F(x) \) is a function with period \( p \) it is well known that

\[
\sum_{z=1}^{p-1} F(x) + \sum_{z=1}^{p-1} \left( \frac{x}{p} \right) F(x) = \sum_{z=1}^{p-1} F(x^z),
\]

(1)

and, if \( a \equiv 0 \pmod{p} \), Jacobsthal [1] has noted that

\[
\sum_{z=0}^{p-1} F(x) + \sum_{z=0}^{p-1} \left( \frac{x^2 - 4a}{p} \right) F(x) = \sum_{z=1}^{p-1} F(x + a\bar{x}),
\]

(2)

where \( \left( \frac{x}{p} \right) \) is the Legendre symbol and \( \bar{x} \) is the unique integer such that \( x\bar{x} \equiv 1 \pmod{p} \) and \( 0 < \bar{x} < p \). It is the purpose of this note to prove a result of a similar type to (1) and (2), namely,

\[
\sum_{z=0}^{p-1} \left( \frac{x - 2}{p} \right) F(x) + \sum_{z=0}^{p-1} \left( \frac{x + 2}{p} \right) F(x) = \sum_{z=0}^{p-1} \left( \frac{x}{p} \right) F(x + \bar{x}).
\]

(3)

The value of the Salié sum [3] can be deduced from (3) since with the obvious choice of \( F(x) \) we obtain the Salié sum as the sum of two Gaussian sums.

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Taking \( a = 1 \) in (2) and replacing \( F(x) \) by \( \left( \frac{x - 2}{p} \right) F(x) \) we obtain
\[
\sum_{x=0}^{p-1} \left( \frac{x - 2}{p} \right) F(x) + \sum_{x=0}^{p-1} \left( \frac{x + 2}{p} \right) \left( \frac{(x - 2)^2}{p} \right) F(x) = \sum_{x=1}^{p-1} \left( \frac{x - 2 + \bar{x}}{p} \right) F(x + \bar{x}),
\]
i.e.,
\[
\sum_{x=0}^{p-1} \left( \frac{x - 2}{p} \right) F(x) + \sum_{\substack{x=0 \\ x \neq 2}}^{p-1} \left( \frac{x + 2}{p} \right) F(x) = \sum_{x=1}^{p-1} \left( \frac{x}{p} \right) F(x + \bar{x}).
\]
(4)

This proves (3) as \( \left( \frac{4}{p} \right) F(2) = \left( \frac{1}{p} \right) F(1 + \bar{1}) \).

Taking \( F(x) = e(kx) (k \neq 0) \) in (3), where \( e(t) = e^{2\pi it/p} \), we obtain
\[
\sum_{x=1}^{p-1} \left( \frac{x}{p} \right) e(k(x + \bar{x})) = \sum_{x=0}^{p-1} \left( \frac{x - 2}{p} \right) e(kx) + \sum_{\substack{x=0 \\ x \neq 2}}^{p-1} \left( \frac{x + 2}{p} \right) e(kx),
\]
which gives Salié's result [3]
\[
\sum_{x=1}^{p-1} \left( \frac{x}{p} \right) e(k(x + \bar{x})) = \left( \frac{k}{p} \right) i^{\frac{1}{2}(p-1)} p^\frac{1}{2} (e(2k) + e(-2k)),
\]
in view of the Gaussian sum [2]
\[
\sum_{x=0}^{p-1} \left( \frac{x - a}{p} \right) e(kx) = \left( \frac{k}{p} \right) i^{\frac{1}{2}(p-1)} p^\frac{1}{2} e(ka).
\]

REFERENCES