ASYMPTOTIC BEHAVIOUR OF THE $n^{th}$ TERM OF CERTAIN SUBSEQUENCES OF THE NATURAL NUMBERS

KENNETH S. WILLIAMS
(Carleton University, Ottawa)

Let $A = \{a(1), a(2), \ldots, a(n), \ldots\}$ be an infinite subsequence of the natural numbers. Number theory provides us with a wealth of examples of such subsequences $A$ for which the asymptotic behaviour is known (as $x \to +\infty$) of the number $\pi_A(x)$ of elements in $A$, which are less than or equal to the real number $x$. (For a few such examples see [2]—[12]). For example [3] if $A$ is the subsequence of squarefree integers it is known that

$$\pi_A(x) = \frac{6}{\pi^2} x + O(x^{\frac{1}{2}}) \text{ as } x \to +\infty.$$ 

Hence as $\pi_A(a(n)) = n$, we have

$$n = \frac{6}{\pi^2} a(n) + O((a(n))^{\frac{1}{2}}),$$

from which we deduce

$$a(n) = O(n),$$

and so

$$n = \frac{6}{\pi^2} a(n) + O(n^{\frac{1}{2}}).$$

that is,

$$a(n) = \frac{\pi^2}{6} n + O(n^{\frac{1}{2}}), \text{ as } n \to +\infty.$$
Thus we have deduced the asymptotic behaviour of \( a(n) \) from the known asymptotic behaviour of \( \pi_A(x) \). It is the purpose of this paper to do this for a general subsequence \( A \) for which the asymptotic behaviour of \( \pi_A(x) \) is known. We suppose that an asymptotic formula for \( \pi_A(x) \) is known of the following type:

\[
\pi_A(x) = f(x) + O(g(x)), \quad \text{as} \quad x \to +\infty,
\]

where the constant implied by the \( O \)-symbol is independent of \( x \). It will always be understood that such an expression as (1) is a genuine asymptotic formula, that is, \( f(x) \) is the "main term", so that \( \pi_A(x) \sim f(x) \), as \( x \to +\infty \), and \( O(g(x)) \) is the "error term". This is guaranteed by

\[
\lim_{x \to +\infty} \frac{g(x)}{f(x)} = 0.
\]

We prove

**Theorem.** Let \( A = \{a(1), a(2), \ldots\} \) be an infinite subsequence of the natural numbers for which an asymptotic formula (1) is known, where for all sufficiently large \( x \), \( f'(x) \) and \( g'(x) \) both exist, and satisfy

\[
(2) \quad f'(x) > O, \quad g'(x) > O, \quad \frac{f(x)}{f'(x)} = O(x), \quad \frac{g'(x)}{g(x)} = O \left( \frac{1}{x} \right),
\]

as \( x \to +\infty \).

Then if \((k(x), h(x))\) is a pair of real-valued functions with \( h(x) = o(x) \), as \( x \to +\infty \), such that for all sufficiently large \( x \) we have

\[
(3) \quad f(k(x)) = x + O(h(x)),
\]

then

\[
(4) \quad a(n) = k(n) + O \left( \frac{k(n)}{n} \max \left( g(k(n)), h(n) \right) \right),
\]

as \( n \to +\infty \).
We note that (2) implies the existence of $f^{-1}(x)$ for all sufficiently large $x$ and so there is always a pair $(k(x), h(x))$ satisfying (3), namely $(k(x), h(x)) = (f^{-1}(x), 0)$. With this choice the theorem gives

$$(5) \quad a(n) = f^{-1}(n) + O\left(\frac{f^{-1}(n) g(f^{-1}(n))}{n}\right), \text{ as } n \to \infty.$$ 

However as we shall see in the examples concluding this paper, it is often more convenient to apply (4) with $k(x) \neq f^{-1}(x)$ rather than (5).

In the proof of the theorem we make use of the following theorem due to Entringer [1], namely, if $r(x) \to +\infty$ and $r(x) \sim s(x)$ as $x \to +\infty$, and $t(x)$ is monotonic and

$$\frac{t'(x)}{t(x)} = O\left(\frac{1}{x}\right)$$

for all sufficiently large $x$, then $t(r(x)) \sim t(s(x))$, as $x \to +\infty$.

**Proof of Theorem.** As $f'(x) > 0$, for all sufficiently large $x$, $f^{-1}(x)$ exists and is differentiable with positive derivative

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}, \text{ for all sufficiently large } x. \quad \text{Moreover as }$$

$A$ is an infinite subsequence we have $\pi_A(x) \to +\infty$, as $x \to +\infty$. But $\pi_A(x) \sim f(x)$, as $x \to +\infty$, so we must have $f(x) \to +\infty$, as $x \to +\infty$. Thus $f^{-1}(x) \to +\infty$, as $x \to +\infty$, and so choosing

$$y = f^{-1}(x) \text{ in } \frac{f(y)}{f'(y)} = O(y), \text{ as } y \to +\infty, \text{ we obtain }$$

$$\frac{(f^{-1})'(x)}{f^{-1}(x)} = \frac{1}{xf^{-1}(x)} \cdot \frac{f(f^{-1}(x))}{f'(f^{-1}(x))}$$

$$= \frac{1}{xf^{-1}(x)} \cdot O(f^{-1}(x)) = O\left(\frac{1}{x}\right),$$

as $x \to +\infty$. Now from (3), as $h(x) = o(x)$, as $x \to +\infty$, we have

$$f(k(x)) \sim x, \text{ as } x \to +\infty.$$
Thus for all sufficiently large $x$, we have

\begin{equation}
(8) \quad f(k(x)) \geq \frac{x}{2}.
\end{equation}

From (6) and (7) by Entringer's theorem we have

\begin{equation}
(9) \quad k(x) = f^{-1}(f(k(x))) \sim f^{-1}(x), \text{ as } x \to +\infty, \text{ and so for all sufficiently large } x \text{ we have}
\end{equation}

\begin{equation}
(10) \quad \frac{1}{2} k(x) \leq f^{-1}(x) \leq \frac{3}{2} k(x).
\end{equation}

From (9) we deduce $k(x) \to +\infty$, as $x \to +\infty$, and so as $f(x)$ is monotonic increasing we have from (10) for all sufficiently large $x$

\begin{equation}
(11) \quad x \leq f\left(\frac{3}{2} k(x)\right).
\end{equation}

Hence from (8) and (11) we have for all sufficiently large $x$

\begin{equation}
(12) \quad \max (x, f(k(x))) \leq f\left(\frac{3}{2} k(x)\right) \min (x, f(k(x))) \geq \frac{x}{2}.
\end{equation}

Now by the mean value theorem there exists $c(x)$ satisfying $\min (x, f(k(x))) \leq c(x) \leq \max (x, f(k(x)))$ and such that

\begin{equation}
(13) \quad | f^{-1}(x) - k(x) | = | f^{-1}(x) - f^{-1}(f(k(x))) | = | (f^{-1})'(c(x))(x - f(k(x))) |.
\end{equation}

From (12) we deduce that

\begin{equation}
(14) \quad \frac{x}{2} \leq c(x) \leq f\left(\frac{3}{2} k(x)\right).
\end{equation}

and so as $f^{-1}(x)$ is monotonic increasing for all sufficiently large $x$, we have

\[ f^{-1}(c(x)) \leq \frac{3}{2} k(x). \]
Hence from (6) and (14) we have

\[ (f^{-1})'(c(x)) = O\left( \frac{f^{-1}(c(x))}{c(x)} \right) = O\left( \frac{k(x)}{x} \right), \]

as \( x \to +\infty \).

Thus from (3), (13) and (15) we deduce

\[ |f^{-1}(x) - k(x)| = O\left( \frac{k(x)h(x)}{x} \right), \quad \text{as} \quad x \to +\infty. \]

Now taking \( x = a(n), \ n \to +\infty \), in (1) we obtain

\[ n = \pi_A(a(n)) = f(a(n)) + O(g(a(n))), \]

as \( n \to +\infty \), that is,

\[ n \sim f(a(n)), \quad \text{as} \quad n \to +\infty, \]

and so in particular for all sufficiently large \( n \) we have

\[ f(a(n)) \geq \frac{n}{2}. \]

From (18) by Entringer's theorem we have

\[ f^{-1}(u) \sim a(n), \quad \text{as} \quad n \to \infty, \]

and so in particular for all sufficiently large \( n \) we have

\[ \frac{1}{2} a(n) \leq f^{-1}(u) \leq \frac{3}{2} a(n). \]

Thus as \( f(x) \) is increasing for all sufficiently large \( x \), and \( a(n) \to +\infty \), as \( n \to +\infty \), we deduce from (21) that for all sufficiently large \( n \),

\[ n \leq f\left( \frac{3}{2} a(n) \right). \]

Hence from (19) and (22) we have for all sufficiently large \( n \)
max \( (n, f(a(n))) \leq f \left( \frac{3}{2} a(n) \right) \), \\
(23) \quad \min \ (n, f(a(n))) \geq \frac{n}{2}.

Now by the mean value theorem there exists \(d(n)\) satisfying \\
\(\min (n, f(a(n))) \leq d(n) \leq \max (n, f(a(n)))\) such that \\
(24) \quad |a(n) - f^{-1}(n)| = |f^{-1}(f(a(n))) - f^{-1}(n)| \\
= |(f^{-1})'(d(n))(f(a(n)) - n)|.

From (23) we deduce that for all sufficiently large \(n\) \\
(25) \quad \frac{n}{2} \leq d(n) \leq f \left( \frac{3}{2} a(n) \right),

and so as \(f^{-1}(x)\) is monotonic increasing for all sufficiently large \(x\), we have \\
\(f^{-1}(d(n)) \leq \frac{3}{2} a(n)\).

Hence from (6) and (25) we have \\
(26) \quad (f^{-1})'(d(n)) = O \left( \frac{f^{-1}(d(n))}{d(n)} \right) = O \left( \frac{a(n)}{n} \right), \text{ as } n \to +\infty.

Moreover from (9) and (20) we have \\
(27) \quad a(n) \sim f^{-1}(n) \sim k(n), \text{ as } n \to +\infty,

so that (26) becomes \\
(28) \quad (f^{-1})'(d(n)) = O \left( \frac{k(n)}{n} \right), \text{ as } n \to +\infty.

From (2), \(g(x)\) satisfies the conditions of Entringer’s theorem and so by (27) we can deduce \\
(29) \quad g(a(n)) \sim g(k(n)), \text{ as } n \to +\infty,

so that from (17), (24), (28), (29) we obtain
(30) \[ |a(n) - f^{-1}(n)| = O \left( \frac{k(n)g(k(n))}{n} \right), \text{ as } n \to +\infty. \]

(4) now follows from (16) and (30) in view of the inequality

\[ |a(n) - k(n)| \leq |a(n) - f^{-1}(n)| + |f^{-1}(n) - k(n)|. \]

We remark that (4) is a genuine asymptotic formula as \( h(n) = o(n) \) and \( g(k(n)) = o(f(k(n)) = o(n), \text{ as } n \to +\infty. \)

We conclude this paper with two examples.

**Example 1.** Let \( a(n) = n^{th} \) integer which is the sum of two squares. Then it is known [5] (p. 261) that

\[
\pi_A(x) = \frac{Bx}{\log^\frac{3}{2} x} + O \left( \frac{x}{\log^\frac{8}{3} x} \right), \text{ as } x \to +\infty,
\]

where

\[ B = \frac{1}{\sqrt{2}} \prod_{r = 1}^{\infty} \left( 1 - \frac{1}{r^2} \right)^{-\frac{1}{2}}. \]

Thus we may take

\[ f(x) = \frac{Bx}{\log^\frac{3}{2} x} \text{ and } g(x) = \frac{x}{\log^\frac{8}{3} x}. \]

It is easily verified that the conditions given in (2) are satisfied. Further as (see below)

\[ f\left( \frac{x \log^\frac{3}{2} x}{B} \right) = x + o \left( \frac{x \log \log x}{\log x} \right), \text{ as } x \to +\infty. \]

we can choose

\[ (k(x), h(x)) = \left( \frac{x \log^\frac{4}{3} x}{B}, \frac{x \log \log x}{\log x} \right). \]
Then by the theorem we have

$$u(n) = \frac{n \log^\frac{1}{2} n}{B} + O(n \log^\frac{1}{2} n), \text{ as } n \to +\infty.$$  

**Proof of (31).** For $x \geq \exp(B^\frac{1}{2})$.

so that

$$\frac{\log^\frac{1}{2} x}{B} \geq 1,$$

we have

$$x - f\left(\frac{x \log^\frac{1}{2} x}{B}\right) = x - x\left(\log \left(\frac{1}{B} x \log^\frac{1}{2} x\right)\right)^{\frac{1}{2}}$$

$$\geq x - x\left(\frac{\log x}{\log x}\right)^{\frac{1}{2}} = 0,$$

so that as $x \to +\infty$ we have

$$|x - f\left(\frac{x \log^\frac{1}{2} x}{B}\right)| = x \cdot f\left(\frac{x \log^\frac{1}{2} x}{B}\right)$$

$$= x \left\{\log^\frac{1}{2}\left(\frac{1}{B} x \log^\frac{1}{2} x\right) - \log^\frac{1}{2} x\right\}$$

$$\leq \frac{x}{\log^\frac{1}{2} x} \left\{\log^\frac{1}{2}\left(\frac{1}{B} x \log^\frac{1}{2} x\right) - \log^\frac{1}{2} x\right\}$$

$$= x \left[\left(1 + \frac{\log \left(\frac{\log^\frac{1}{2} x}{B}\right)}{\log x}\right)^{\frac{1}{2}} - 1\right]$$
\[ \leq \frac{x}{2} \log \left( \frac{\log \frac{1}{2} x}{B} \right) \log x \]
\[ = O \left( \frac{x \log \log x}{\log x} \right) \]

as required.

**Example 2.** Let \( a(n) = n^{th} \) prime number. It is well-known [5] (p. 250) that one form of the prime number theorem with error term is

\[ \Pi_A(x) = li(x) + O(x e^{-c\sqrt{\log x}}), \text{ as } x \to +\infty, \]

where \( c \) is a positive constant and \( li(x) \) is the logarithmic integral

\[ \int_{2}^{x} \frac{dt}{\log t}. \]

Thus we may take \( f(x) = li(x) \) and \( g(x) = xe^{-c\sqrt{\log x}} \). It is easily verified that the conditions given in (2) are satisfied. Further as (see below).

\[ li(x \log x) = x + O \left( \frac{x \log \log x}{\log x} \right), \text{ as } x \to +\infty, \]

we can choose

\[ (k(x), h(x)) = \left( x \log x, \frac{x \log \log x}{\log x} \right). \]

Then by the theorem we have

\[ a(n) = n \log n + O \left( n \log \log n \right), \text{ as } n \to +\infty. \]

**Proof of (32).** We have on integrating by parts

\[ li \left( x \log x \right) - x = \int_{2}^{x} \frac{\log x}{\log t} dt - x. \]
\[
\frac{-x \log \log x}{\log (x \log x)} + \int_2^x \frac{\log t}{\log t} dt + O(1)
\]

\[
= O \left( \frac{x \log \log x}{\log x} \right),
\]

as

\[
\int_2^x \frac{\log x}{\log^2 t} dt = \left( \int_2^\sqrt{x} \frac{x \log x}{\log^2 t} dt \right) + \left( \int_{\sqrt{x}}^x \frac{x \log x}{\log^2 t} dt \right)
\]

\[
= O(\sqrt{x}) + O \left( \frac{x \log x - \sqrt{x}}{\log^2 \sqrt{x}} \right)
\]

\[
= O \left( \frac{x}{\log x} \right).
\]

REFERENCES


9. **H. Onishi**, The number of positive integers $n \leq N$ such that $n, n+a_2, n+a_3, \ldots, n+a_r$ are all square-free, Jour. Lond. Math. Soc., 41 (1966), 138-140.

