INTEGERS OF BIQUADRATIC FIELDS

BY

KENNETH S. WILLIAMS(*)

Let $Q$ denote the field of rational numbers. If $m$, $n$ are distinct squarefree integers the field formed by adjoining $\sqrt{m}$ and $\sqrt{n}$ to $Q$ is denoted by $Q(\sqrt{m}, \sqrt{n})$. Since $Q(\sqrt{m}, \sqrt{n}) = Q(\sqrt{m} + \sqrt{n})$ and $\sqrt{m} + \sqrt{n}$ has for its unique minimal polynomial $x^4 - 2(m+n)x^2 + (m-n)^2$, $Q(\sqrt{m}, \sqrt{n})$ is a biquadratic field over $Q$. The elements of $Q(\sqrt{m}, \sqrt{n})$ are of the form $a_0 + a_1 \sqrt{m} + a_2 \sqrt{n} + a_3 \sqrt{mn}$, where $a_0, a_1, a_2, a_3 \in Q$. Any element of $Q(\sqrt{m}, \sqrt{n})$ which satisfies a monic equation of degree $\geq 1$ with rational integral coefficients is called an integer of $Q(\sqrt{m}, \sqrt{n})$. The set of all these integers is an integral domain. In this paper we determine the explicit form of the integers of $Q(\sqrt{m}, \sqrt{n})$ (Theorem 1), an integral basis for $Q(\sqrt{m}, \sqrt{n})$ (Theorem 2), and the discriminant of $Q(\sqrt{m}, \sqrt{n})$ (Theorem 3). (With $Q(\sqrt{m}, \sqrt{n})$ considered as a relative quadratic field, that is, as a quadratic field over $Q(\sqrt{m})$, an integral basis for $Q(\sqrt{m}, \sqrt{n})$ has been given in [1].)

The form of the integers of a quadratic field are well known [3]. If $k$ is a squarefree integer then the integers of $Q(\sqrt{k})$ are given by $\frac{1}{2}(x_0 + x_1 \sqrt{k})$, where $x_0, x_1$ are integers such that $x_0 \equiv x_1 \pmod{2}$, if $k \equiv 1 \pmod{4}$; and by $x_0 + x_1 \sqrt{k}$, where $x_0, x_1$ are integers, if $k \equiv 2$ or $3 \pmod{4}$. Thus we know the integers of the subfields $Q(\sqrt{m}), Q(\sqrt{n}), Q(\sqrt{mn})$ of $Q(\sqrt{m}, \sqrt{n})$.

We begin by making some simplifying assumptions about $m$ and $n$. We let $l = (m, n)$ and write $m = ln_1, n = ln_1$ so that $(n_1, n_1) = 1$. Since $m, n$ are squarefree we have the following possibilities for the residues of $m, n, m_1n_1$ modulo 4.

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Thus as
\[ Q(\sqrt{m}, \sqrt{n}) = Q(\sqrt{m}, \sqrt{m_1 n_1}) = Q(\sqrt{n}, \sqrt{m_1 n_1}) = Q(\sqrt{n}, \sqrt{m}) \]
we may suppose without loss of generality that
\[ (m, n) \equiv (1, 1), (1, 2), (2, 3) \text{ or } (3, 3) \pmod{4}. \]

We now determine the form of the integers of \( Q(\sqrt{m}, \sqrt{n}) \), where \( m, n \) satisfy (1).

**Theorem 1.** Letting \( x_0, x_1, x_2, x_3 \) denote rational integers, the integers of \( Q(\sqrt{m}, \sqrt{n}) \) are given as follows:

(i) if \((m, n) \equiv (m_1, n_1) \equiv (1, 1) \pmod{4}\), the integers are
\[ \frac{1}{4}(x_0 + x_1 \sqrt{m + x_2 \sqrt{n + x_3 \sqrt{m_1 n_1}}}), \]
where \( x_0 \equiv x_1 \equiv x_2 \equiv x_3 \pmod{2} \), \( x_0 - x_1 + x_2 - x_3 \equiv 0 \pmod{4} \);

(ii) if \((m, n) \equiv (1, 1), (m_1, n_1) \equiv (3, 3) \pmod{4}\), the integers are
\[ \frac{1}{4}(x_0 + x_1 \sqrt{m + x_2 \sqrt{n + x_3 \sqrt{m_1 n_1}}}), \]
where \( x_0 \equiv x_1 \equiv x_2 \equiv x_3 \pmod{2} \), \( x_0 - x_1 - x_2 - x_3 \equiv 0 \pmod{4} \);

(iii) if \((m, n) \equiv (1, 2) \pmod{4}\), the integers are
\[ \frac{1}{4}(x_0 + x_1 \sqrt{m + x_2 \sqrt{n + x_3 \sqrt{m_1 n_1}}}), \]
where \( x_0 \equiv x_1, x_2 \equiv x_3 \pmod{2} \);

(iv) if \((m, n) \equiv (2, 3) \pmod{4}\), the integers are
\[ \frac{1}{4}(x_0 + x_1 \sqrt{m + x_2 \sqrt{n + x_3 \sqrt{m_1 n_1}}}), \]
where \( x_0 \equiv x_2 \equiv 0, x_1 \equiv x_3 \pmod{2} \);

(v) if \((m, n) \equiv (3, 3) \pmod{4}\), the integers are
\[ \frac{1}{4}(x_0 + x_1 \sqrt{m + x_2 \sqrt{n + x_3 \sqrt{m_1 n_1}}}), \]
where \( x_0 \equiv x_3, x_1 \equiv x_2 \pmod{2} \).

**Proof.** Let \( \theta \) be an integer of \( Q(\sqrt{m}, \sqrt{n}) \), where \( m, n \) satisfy (1). Then \( \theta \) can be written
\[ \theta = a_0 + a_1 \sqrt{m} + a_2 \sqrt{n} + a_3 \sqrt{m_1 n_1}, \]
where \( a_0, a_1, a_2, a_3 \in Q \). As \( \theta \) is an integer of \( Q(\sqrt{m}, \sqrt{n}) \) so are its conjugates over \( Q \), namely,
\[ \begin{align*}
\theta' &= a_0 + a_1 \sqrt{m} - a_2 \sqrt{n} - a_3 \sqrt{m_1 n_1}, \\
\theta'' &= a_0 - a_1 \sqrt{m} + a_2 \sqrt{n} - a_3 \sqrt{m_1 n_1}, \\
\theta''' &= a_0 - a_1 \sqrt{m} - a_2 \sqrt{n} + a_3 \sqrt{m_1 n_1}.
\end{align*} \]
The three quantities
\[
\begin{align*}
\theta + \theta' &= 2a_0 + 2a_1 \sqrt{m} \in Q(\sqrt{m}), \\
\theta + \theta^* &= 2a_0 + 2a_2 \sqrt{n} \in Q(\sqrt{n}), \\
\theta + \theta'' &= 2a_0 + 2a_3 \sqrt{m_1n_1} \in Q(\sqrt{m_1n_1}),
\end{align*}
\]
are therefore all integers of \(Q(\sqrt{m}, \sqrt{n})\). Hence they must be integers of \(Q(\sqrt{m})\), \(Q(\sqrt{n})\), \(Q(\sqrt{m_1n_1})\) respectively.

We consider the cases \((m, n)\equiv(1, 2), (2, 3), (3, 3) \pmod{4}\) first so that at least two of \(m, n, m_1n_1\) are not congruent to 1 \(\pmod{4}\), and so at least two of (4) have integral coefficients. Since \(2a_0\) is common to all three of (4), the third one must also have integral coefficients. Hence \(2a_0, 2a_1, 2a_2, 2a_3\) are all integers and we can write (2) as
\[
\theta = \frac{1}{2}(b_0 + b_1 \sqrt{m} + b_2 \sqrt{n} + b_3 \sqrt{m_1n_1}),
\]
where \(b_0, b_1, b_2, b_3\) are all integers. Let us define
\[
\begin{align*}
\epsilon &= b_0^2 - m_1n_1b_3^2, \\
d &= b_0^2 - mb_1^2 - nb_2^2 + m_1n_1b_3^2, \\
e &= 2(b_0b_3 - b_1b_2),
\end{align*}
\]
so that \(\theta\) satisfies
\[
\theta^4 - 2b_0\theta^3 + \left(c + \frac{d}{2}\right)\theta^2 + \frac{b_0^2m_1n_1e - b_0d}{2} + \frac{(d^2 - m_1n_1e^2)}{16} = 0.
\]
If \(\theta \in Q(\sqrt{m}), Q(\sqrt{n})\) or \(Q(\sqrt{m_1n_1})\) the theorem is easily verified so we suppose that \(\theta \notin Q(\sqrt{m}), Q(\sqrt{n}), Q(\sqrt{m_1n_1})\). Thus the coefficients of (7) must all be integers, that is, we must have
\[
d^2 - m_1n_1e^2 \equiv 0 \pmod{16},
\]
since as \(e\) is even this implies that \(d\) must be even too.

If \((m, n)\equiv(1, 2) \pmod{4}\), so that \(l \equiv 1 \pmod{2}\), \(m_1n_1 \equiv 2 \pmod{4}\), (8) is equivalent to \(d \equiv e \equiv 0 \pmod{4}\), or
\[
\begin{align*}
&b_0^2 - b_1^2 - 2b_2^2 + 2b_3^2 \equiv 0 \pmod{4}, \\
&b_0b_3 - b_1b_2 \equiv 0 \pmod{2}.
\end{align*}
\]
If \(b_0 \neq b_1 \pmod{2}\) then \(b_0^2 - b_1^2 \equiv 1 \pmod{2}\) and (9a) is insoluble. Thus we must have \(b_0 \equiv b_1 \pmod{2}\), so \(b_0^2 - b_1^2 \equiv 0 \pmod{4}\) and (9a) implies \(2(b_2^2 - b_3^2) \equiv 0 \pmod{4}\), that is \(b_2 \equiv b_3 \pmod{2}\). Clearly (9b) is then satisfied and this proves case (iii) of the theorem.

If \((m, n)\equiv(2, 3) \pmod{4}\), so that \(l \equiv 1 \pmod{2}\), \(m_1n_1 \equiv 2 \pmod{4}\), (8) is equivalent to \(d \equiv e \equiv 0 \pmod{4}\), or
\[
\begin{align*}
&b_0^2 - 2b_1^2 + b_2^2 + 2b_3^2 \equiv 0 \pmod{4}, \\
&b_0b_3 - b_1b_2 \equiv 0 \pmod{2}.
\end{align*}
\]
If either \( b_0 \) or \( b_2 \) is odd (10a) implies that the other is odd too. Then (10b) implies \( b_1 \equiv b_3 \pmod{2} \) and (10a) becomes \( 1 - 2b_1^2 + 1 + 2b_3^2 \equiv 0 \pmod{4} \), which is impossible. Thus \( b_0 \equiv b_2 \equiv 0 \pmod{2} \) and so \( b_1 \equiv b_3 \pmod{2} \). This proves case (iv) of the theorem.

If \( (m, n) \equiv (3, 3) \pmod{4} \), so that \( j \equiv 1 \pmod{2} \), \( m_1n_1 \equiv 1 \pmod{4} \), (8) is equivalent to \( d \equiv e \pmod{4} \), or

\[
b_0^2 + b_1^2 + b_2^2 + b_3^2 \equiv 2(b_0b_3 - b_1b_2) \pmod{4},
\]
or

\[
(b_0 - b_3)^2 + (b_1 + b_2)^2 \equiv 0 \pmod{4}.
\]

Thus we have \( b_0 \equiv b_3 \), \( b_1 \equiv b_2 \pmod{2} \), which proves case (v) of the theorem.

We now consider the case \( (m, n) \equiv (1, 1) \pmod{4} \), which has been excluded up to this point. We have \( m_3n_1 \equiv 1 \pmod{4} \) so that \( 2a_0, 2a_1, 2a_2, 2a_3 \) are either all integers or all halves of odd integers.

If \( 2a_0, 2a_1, 2a_2, 2a_3 \) are all integers then as in the case \( (m, n) \equiv (3, 3) \pmod{4} \) we have \( d \equiv e \pmod{4} \), that is,

\[
b_0^2 + b_1^2 - b_2^2 + b_3^2 \equiv 2(b_0b_3 - b_1b_2) \pmod{4},
\]
or

\[
(b_0 - b_3)^2 - (b_1 - b_2)^2 \equiv 0 \pmod{4},
\]
which implies

\[
b_0 - b_3 \equiv b_1 - b_2 \pmod{2}
\]
or

\[
b_0 - b_3 \equiv b_1 + b_2 - b_0 \equiv 0 \pmod{2}.
\]

This gives \( \theta \) in the form \( \frac{1}{4}(c_0 + c_1 \sqrt{m} + c_2 \sqrt{n} + c_3 \sqrt{m_1n_1}) \), with \( c_0, c_1, c_2, c_3 \) integers such that

\[
c_0 \equiv c_1 \equiv c_2 \equiv c_3 \equiv 0 \pmod{2}, \quad c_0 - c_1 \pm c_2 \mp c_3 \equiv 0 \pmod{4}.
\]

If \( 2a_0, 2a_1, 2a_2, 2a_3 \) are all halves of odd integers we can write (2) as

\[
\theta = \frac{1}{4}(c_0 + c_1 \sqrt{m} + c_2 \sqrt{n} + c_3 \sqrt{m_1n_1}),
\]
where \( c_0, c_1, c_2, c_3 \) are integers such that \( c_0 \equiv c_1 \equiv c_2 \equiv c_3 \equiv 1 \pmod{2} \). We have

\[
c = \frac{c_0^2 - m_1n_1c_3^2}{4}, \quad d = \frac{c_0^2 - m_1n_1c_3^2 - nc_3^2 + m_1n_1c_3^2}{4},
\]

\[
e = \frac{c_0c_3 - c_1c_2}{2}.
\]
These are all integers as \(c_0 \equiv c_1 \equiv c_2 \equiv c_3 \equiv l \equiv 1 \pmod{2}\) and \(m \equiv n \equiv m_1 n_1 \equiv 1 \pmod{4}\). Moreover
\[
c_3^2 - m c_1^2 - n c_2^2 + m_1 n_1 c_3^2 \equiv 1 - m - n + m_1 n_1 \pmod{8}
\equiv 1 - m - n + l^2 m_1 n_1 \pmod{8}
= 1 - m - n + mn
= (1 - m)(1 - n)
\equiv 0 \pmod{8},
\]
so that \(d\) is even. Now \(\theta\) satisfies
\[
\theta^4 - c_0 \theta^3 + \left(\frac{e + d}{2}\right) \theta^2 + \left(\frac{c_2 m_1 n_1 e - c_6 d}{4}\right) \theta + \left(\frac{d^2 - m_1 n_1 e^2}{16}\right) = 0.
\]
Clearly \(\theta \notin Q(\sqrt{m}), Q(\sqrt{n}), Q(\sqrt{m_1 n_1})\) so that the coefficients of (13) must all be integers, that is, we must have
\[
d^2 - m_1 n_1 e^2 \equiv 0 \pmod{16},
\]
(14) since (14) implies, as \(d \equiv 0 \pmod{2}\), \(m_1 n_1 \equiv 1 \pmod{4}\), that \(d \equiv e \pmod{4}\) and so
\[
c_2 m_1 n_1 e - c_6 d \equiv c_2 e - c_6 d \equiv d(c_3 - c_0) \equiv 0 \pmod{4}.
\]
Clearly as \(d \equiv 0 \pmod{2}\), (14) is equivalent to \(d \equiv e \pmod{4}\).

Writing \(c_i = 2d_i + 1 \quad (i = 0, 1, 2, 3)\) we have
\[
d = (d_0^2 - m d_1^2 - n d_2^2 + m_1 n_1 d_3^2) + (d_0 - m d_1 - n d_2 + m_1 n_1 d_3) + \frac{(1 - m - n + m_1 n_1)}{4}
\equiv (d_0^2 - d_1^2 - d_2^2 + d_3^2) + (d_0 - d_1 - d_2 + d_3) + \frac{(1 - m - n + m_1 n_1)}{4} \pmod{4},
\]
and
\[
e = (2d_0 d_3 - 2d_1 d_2) + (d_0 - d_1 - d_2 + d_3) + \frac{1 - l}{2}.
\]
Thus if \(l \equiv 1 \pmod{4}\), so that \((m_1, n_1) \equiv (1, 1) \pmod{4}\), we have
\[
d \equiv (d_0^2 - d_1^2 - d_2^2 + d_3^2) + (d_0 - d_1 - d_2 + d_3) + \frac{1 - l}{2} \pmod{4},
\]
\[
e \equiv (2d_0 d_3 - 2d_1 d_2) + (d_0 - d_1 - d_2 + d_3) + \frac{1 - l}{2} \pmod{4},
\]
and so \(d \equiv e \pmod{4}\) gives
\[
(d_0 - d_3)^2 - (d_1 - d_2)^2 \equiv 0 \pmod{4},
\]
that is
\[
d_0 - d_3 \equiv d_1 - d_2 \pmod{2},
\]
or
\[
c_0 - c_1 + c_2 - c_3 \equiv 0 \pmod{4},
\]
which completes the proof of case (i) of the theorem.
If \( l \equiv 3 \pmod{4} \), so that \((m_1, n_1) \equiv (3, 3) \pmod{4}\), we have
\[
d \equiv (d_0^2 - d_1^2 - d_2^2 + d_3^2) + (d_0 - d_1 - d_2 + d_3) + \frac{1 + l}{2} \pmod{4},
\]
\[
e \equiv (2d_0d_3 + 2d_1d_2) + (d_0 + d_1 + d_2 + d_3) + \frac{1 - l}{2} \pmod{4},
\]
and so \( d \equiv e \pmod{4} \) gives
\[
(d_0 - d_3)^2 - (d_1 + d_2)^2 - 2(d_1 + d_2) - 1 \equiv 0 \pmod{4},
\]
that is,
\[
d_0 - d_3 \equiv d_1 + d_2 + 1 \pmod{2},
\]
or
\[
c_0 - c_1 - c_2 - c_3 \equiv 0 \pmod{4},
\]
which completes the proof of case (ii) of the theorem.

We give three simple examples of Theorem 1.

**Example 1.** \( \theta = \frac{1}{4}(5 + 3\sqrt{5} + \sqrt{13} + 3\sqrt{65}) \) is an integer of \( Q(\sqrt{5}, \sqrt{13}) \). \( \theta \) satisfies \( \theta^4 - 5\theta^3 - 71\theta^2 + 120\theta + 1044 = 0 \).

**Example 2.** \( \theta = \frac{1}{4}(1 + \sqrt{21} + \sqrt{33} - \sqrt{77}) \) is an integer of \( Q(\sqrt{21}, \sqrt{33}) \). \( \theta \) satisfies \( \theta^4 - \theta^3 - 16\theta^2 + 37\theta - 17 = 0 \).

**Example 3.** The integers of \( Q(\sqrt{2}, \sqrt{-1}) \) are of the form \( a_0 + a_1\sqrt{2} + a_2\sqrt{-1} + a_3\sqrt{-2} \), where \( a_0, a_2 \) are both integers and \( a_1, a_3 \) are both integers or both halves of odd integers (see [2] for example).

As a consequence of Theorem 1 we have

**Theorem 2.** An integral basis for \( Q(\sqrt{m}, \sqrt{n}) \) is given by

(i) \( \left\{ \frac{1}{2}, \frac{1 + \sqrt{m}}{2}, \frac{1 + \sqrt{n}}{2}, \frac{1 + \sqrt{m} + \sqrt{n} + \sqrt{m_n}}{4}, \right\}, \) if \( \left( m, n \right) \equiv (1, 1), (m_1, n_1) \equiv (1, 1) \pmod{4}, \)

(ii) \( \left\{ \frac{1}{2}, \frac{1 + \sqrt{m}}{2}, \frac{1 + \sqrt{n}}{2}, \frac{1 + \sqrt{m} - \sqrt{n} + \sqrt{m_n}}{4}, \right\}, \) if \( \left( m, n \right) \equiv (1, 1), (m_1, n_1) \equiv (3, 3) \pmod{4}, \)

(iii) \( \left\{ \frac{1}{2}, \sqrt{m}, \sqrt{n}, \frac{\sqrt{m} + \sqrt{m_n}}{2}, \right\}, \) if \( \left( m, n \right) \equiv (1, 2) \pmod{4}, \)

(iv) \( \left\{ \sqrt{m}, \sqrt{n}, \frac{\sqrt{m} + \sqrt{m_n}}{2}, \right\}, \) if \( \left( m, n \right) \equiv (2, 3) \pmod{4}, \)

(v) \( \left\{ \sqrt{m}, \frac{\sqrt{m} + \sqrt{n}}{2}, \frac{\sqrt{m} + \sqrt{n}}{2}, \right\}, \) if \( \left( m, n \right) \equiv (3, 3) \pmod{4}. \)
Proof. We just give the proof of (i) since the other four cases are very similar. By Theorem 1 the general integer of \( Q(\sqrt{m}, \sqrt{n}) \) can be written \( \frac{1}{4}(x_0 + x_1 \sqrt{m} + x_2 \sqrt{n} + x_3 \sqrt{m_1 n_1}) \), where \( x_0, x_1, x_2, x_3 \) are integers such that
\[
x_0 \equiv x_1 \equiv x_2 \equiv x_3 \pmod{2}, \quad x_0 - x_1 + x_2 - x_3 \equiv 0 \pmod{4}.
\]
Write \( z_3 = x_0 \). As \( x_0 \equiv x_1 \equiv x_2 \equiv z_3 \pmod{2} \) there are integers \( y, z_1, z_2, \) such that
\[
x_0 = z_3 + 2y, \quad x_1 = z_3 + 2z_1, \quad x_2 = z_3 + 2z_2.
\]
But as \( x_0 - x_1 + x_2 - z_3 \equiv 0 \pmod{4} \) we have \( y \equiv z_1 + z_2 \pmod{2} \), so there is an integer \( z_0 \) such that \( y = 2z_0 + z_1 + z_2 \). Hence
\[
\frac{1}{4}(x_0 + x_1 \sqrt{m} + x_2 \sqrt{n} + x_3 \sqrt{m_1 n_1})
\]
\[
= z_0 + z_1 \left( \frac{1 + \sqrt{m}}{2} \right) + z_2 \left( \frac{1 + \sqrt{n}}{2} \right) + z_3 \left( \frac{1 + \sqrt{m} + \sqrt{n} + \sqrt{m_1 n_1}}{4} \right),
\]
which proves the result as
\[
1, \quad \frac{1 + \sqrt{m}}{2}, \quad \frac{1 + \sqrt{n}}{2}, \quad \frac{1 + \sqrt{m} + \sqrt{n} + \sqrt{m_1 n_1}}{4},
\]
are integers of \( Q(\sqrt{m}, \sqrt{n}) \).

We illustrate Theorem 2 with a simple example.

Example 4. An integral basis for \( Q(\sqrt{5}, \sqrt{13}) \) is
\[
\{a_0, a_1, a_2, a_3\} = \left\{ 1, \frac{1 + \sqrt{5}}{2}, \frac{1 + \sqrt{13}}{2}, \frac{1 + \sqrt{5} + \sqrt{13} + \sqrt{65}}{4} \right\}
\]
and the integer \( \frac{1}{4}(5 + 3\sqrt{5} + \sqrt{13} + 3\sqrt{65}) \) is given in terms of this integral basis as \( a_0 - a_2 + 3a_3 \).

Finally as the discriminant of an algebraic number field is just the discriminant of an integral basis of the field, we have

Theorem 3. The discriminant of \( Q(\sqrt{m}, \sqrt{n}) \) is given by
\[
(i) \ 16m^2n_1^2, \quad \text{if } (m, n) \equiv (1, 1) \pmod{4},
(ii) \ 16l^2m^2n_1^2, \quad \text{if } (m, n) \equiv (1, 2), \text{ or } (3, 3) \pmod{4},
(iii) \ 64l^2m^2n_1^2, \quad \text{if } (m, n) \equiv (2, 3) \pmod{4}.
\]
Thus, for example, we have

Example 5. The discriminant of \( Q(\sqrt{2}, \sqrt{-1}) \) is 256.
REFERENCES


Carleton University,
Ottawa, Ontario