A SUM OF FRACTIONAL PARTS

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Let \( L \) denote the set of points \( x = (x_1, \cdots, x_n) \) with integral coordinates in Euclidean \( n \)-space. For any prime \( p \geq 3 \), let \( C = C(p) \) be the set of points of \( L \) in the cube \( 0 \leq x_i < p \) \((i = 1, 2, \cdots, n)\). Suppose that \( f(x) \) is any polynomial of degree \( d \) in \( x_1, \cdots, x_n \) with integral coefficients, which does not vanish identically \( (\text{mod } p) \). We let \( \{a\} \) denote the fractional part of the real number \( a \) and consider the problem of estimating

\[
\sum_{x \in C} \{f(x)/p\}
\]

for large primes \( p \). When \( d = 1 \) this sum is just \( \frac{1}{2}(p-1)p^{n-1} = \frac{1}{2}p^n + O(p^{n-1}) \). For \( d > 1 \) we prove, using a result of Uchiyama [1], the following theorem.

**Theorem 1.**

\[
\sum_{x \in C} \{f(x)/p\} = \frac{1}{2}p^n + O(p^{n-1/2} \log p),
\]

as \( p \to \infty \), where the constant implied in the \( O \)-symbol depends only upon \( n \) and \( d \).

**Proof.**

(1)

\[
\sum_{x \in C} \{f(x)/p\} = \sum_{x \in C} \sum_{r=0}^{p-1} \frac{r}{p} \sum_{x \equiv r \pmod{p}} \frac{1}{f(x)}.
\]

Now let \( e(t) \) denote \( \exp(2\pi itp^{-1}) \) for any real \( t \). Then as

\[
(1/p) \sum_{r=0}^{p-1} e(ar) = \begin{cases} 1 & \text{if } a \equiv 0 \pmod{p} \\ 0 & \text{if } a \not\equiv 0 \pmod{p} \end{cases}
\]

we have

\[
\sum_{x \in C} \{f(x)/p\} = (1/p^n) \sum_{r=0}^{p-1} r \sum_{x \in C} e(f(x) - r) = (1/p^n) \sum_{r=0}^{p-1} \sum_{x \in C} e(vf(x)) \sum_{r=0}^{p-1} re(-rv).
\]

The term corresponding to \( v = 0 \) is just

\[
(1/p^n) \sum_{x \in C} \sum_{r=0}^{p-1} r = \frac{1}{2}(p-1)p^{n-1}.
\]

For \( v \neq 0 \) we have

\[
\sum_{r=0}^{p-1} re(-rv) = -p/(1 - e(-v)),
\]
whence
\[ \left| \sum_{x \in \mathbb{C}} \left\{ \frac{f(x)}{p} \right\} - \frac{1}{2} p^{n-1}(p - 1) \right| = \left| \frac{1}{p} \sum_{v=1}^{p-1} \frac{1}{1 - e(-v)} \sum_{x \in \mathbb{C}} e(vf(x)) \right| \]
\[ \leq \left( \frac{1}{p} \right) \sum_{v=1}^{p-1} \left| 1 - e(-v) \right| \left| \sum_{x \in \mathbb{C}} e(vf(x)) \right|. \]

By a recent result of Uchiyama [1]
\[ \left| \sum_{x \in \mathbb{C}} e(vf(x)) \right| \leq k p^{n-1/2} \]
where the constant \( k \) depends only upon \( n \) and \( d \). Hence we have
\[ \left| \sum_{x \in \mathbb{C}} \left\{ \frac{f(x)}{p} \right\} - \frac{1}{2} p^{n-1}(p - 1) \right| \leq \frac{k}{2} p^{n-2/2} \sum_{v=1}^{p-1} \frac{1}{\sin (\pi v / p)}, \]
from which the theorem follows in view of the well-known result
\[ \sum_{v=1}^{p-1} \frac{1}{\sin (\pi v / p)} \leq p \sum_{v=1}^{(p-1)/2} \frac{1}{v} \leq p \log p. \]

We now show that the error term in (1) can be improved for polynomials \( f(x) \) of certain special types, namely diagonal and quadratic polynomials.

**Theorem 2.** If \( f(x) = a_1 x_1^{k_1} + \cdots + a_n x_n^{k_n} + a_0 \), where each \( k_i \geq 1 \) and \( p \nmid a_1 \cdots a_n \), then, provided \( n \geq 3 \),
\[ \sum_{x \in \mathbb{C}} \left\{ \frac{f(x)}{p} \right\} = \left( p^n/2 \right) + O(p^{n-1}), \]
as \( p \to \infty \), where the constant implied in the \( O \)-symbol depends only upon \( k_1, \cdots, k_n \).

**Theorem 3.** Let \( f(x_1, x_2) \) be a quadratic polynomial which is not a function of only one integral variable and which has a discriminant not divisible by \( p \). Suppose further that the discriminant of \( f_1(x_1, x_2, 0) \) is also not divisible by \( p \), where \( f_1(x_1, x_2, x_3) \equiv x_2^2 f(x_1/x_3, x_2/x_3) \). Then
\[ \sum_{x_1=0}^{p-1} \sum_{x_2=0}^{p-1} \left\{ \frac{f(x_1, x_2)}{p} \right\} = \frac{1}{2} p^2 + O(p), \]
as \( p \to \infty \).

**Proof of Theorem 2.** By (2)
\[ \sum_{x \in \mathbb{C}} \left\{ \frac{f(x)}{p} \right\} = \left( \frac{1}{p} \right) \sum_{r=1}^{p-1} r N_r, \]
where \( N_r \) denotes the number of \( x \in \mathbb{C} \) with \( f(x) \equiv r \) \( (\text{mod } p) \). Weil has shown [2] that
\[ N_r = p^{n-1} + O(p^{n/3}), \]
as \( p \to \infty \), where the constant in the \( O \)-symbol depends only on \( k_1, \ldots, k_n \). The theorem follows at once.

**Proof of Theorem 3.** As is well known, under the conditions stated in the theorem, we may reduce \( f(x_1, x_2) \) by a linear nonsingular transformation \((\text{mod } p)\) to the canonical form
\[ ay_1^2 + by_2^2 + c \quad (p \nmid abc). \]
This transformation will not affect the sum under consideration and so we need only consider
\[ \sum_{v_1=0}^{v_{-1}} \sum_{v_2=0}^{v_{-1}} \left( \frac{ay_1^2 + by_2^2 + c}{p} \right) (p \nmid abc). \]
The theorem then follows from (3), as in this case
\[ N_r = \begin{cases} p - (-ab/p) & \text{if } r \neq c \\ p + (p - 1)(-ab/p) & \text{if } r = c. \end{cases} \]

**Conjecture:** (1) may be replaced by
\[ (1') \quad \sum_{x \in \mathbb{C}} \{ f(x)/p \} = \frac{1}{2} p^n + O(p^{n-1/3}), \quad \text{as } p \to \infty. \]

**References**


**TAYLOR'S FORMULA AND THE EXISTENCE OF \( n \)TH DERIVATIVES**

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A common form of Taylor's theorem states sufficient conditions for a function \( f \) to have an expansion of the form
\[ f(a + h) = a_0 + a_1 h + a_2 h^2 + \cdots + a_{n-1} h^{n-1} + R_n(h), \]
valid in some one- or two-sided neighborhood of \( h = 0 \), the remainder term \( R_n \) involving a factor of the form \( f^{(n)}(X) \), where \( X \) is strictly between \( a \) and \( a + h \). The existence (finite or infinite) of \( f^{(n)} \) throughout at least some deleted neighborhood (or one-sided neighborhood) of \( a \) obviously must be postulated in order to prove any theorem of this kind. On the other hand, \( f^{(n)}(a) \) need not exist for the validity of such an expansion.