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AN ELEMENTARY NUMBER-THEORETIC FORMULA

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It is well-known [1] that

$$\sum_{x=0}^{p-1} \left\{ \frac{ax+b}{p} \right\} = p'$$

where p is an odd prime not dividing $a, p' = \frac{1}{2}(p-1)$ and $\{x\}$ denotes the fractional part of x. I wondered if there is a similar formula for

$$\sum_{x=0}^{p-1} \left\{ \frac{ax^4+bx+c}{p} \right\}.$$

I find a formula for this sum, which depends on a sum of Legendre symbols over an incomplete residue system and also on the class number h(-p) if $p \equiv 3 \pmod{4}$. I first prove a simple lemma.

Lemma. If n is a positive integer then

$$\sum_{x=0}^{p-1} x\left(\frac{x+n}{p}\right) - p \sum_{x=0}^{n-1} \left(\frac{x}{p}\right)$$
$$= \begin{cases} 0 \qquad p \equiv 1 \pmod{4} \\ -\frac{p}{3} \left(2 + \left(\frac{2}{p}\right)h(-p) \ p \equiv 3 \pmod{4} \right) \end{cases}$$

Proof. For $n \ge 1$

$$\sum_{x=0}^{p-1} x \left(\frac{x+n}{p} \right) = \sum_{x=1}^{p} (x-1) \left(\frac{x+n-1}{p} \right)$$
$$= \sum_{x=1}^{p} x \left(\frac{x+n-1}{p} \right)$$

since

$$\sum_{x=1}^{p} \left(\frac{x+n-1}{p} \right) = \sum_{y=0}^{p-1} \left(\frac{y}{p} \right) = 0$$

Thus

$$\sum_{x=0}^{p-1} x\left(\frac{x+n}{p}\right) - \sum_{x=0}^{p-1} x\left(\frac{x+n-1}{p}\right) = p\left(\frac{n-1}{p}\right)$$

for $n \ge 1$, yielding

$$\sum_{x=0}^{p-1} x \left(\frac{x+n}{p} \right) = \sum_{x=0}^{p-1} x \left(\frac{x}{p} \right) + p \sum_{x=0}^{p-1} \left(\frac{x}{p} \right)$$

and the lemma follows [2]

$$\sum_{x=0}^{p-1} x\left(\frac{x}{p}\right) = \begin{cases} 0 & p \equiv 1 \pmod{4} \\ -\frac{1}{2} p\left(2 + \left(\frac{2}{p}\right)\right) h\left(-p\right) \\ p \equiv 3 \pmod{4} \end{cases}$$

I can now deduce the formula for the sum in question.

Theorem. If p is an odd prime and p+a then

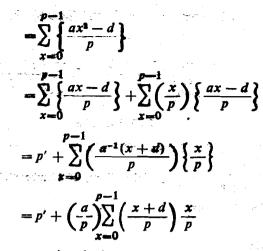
$$\sum_{x=0}^{p-1} \left\{ \frac{ax^3 + bx + c}{p} \right\} = p' + \left(\frac{a}{p}\right) \sum_{x=0}^{d-1} \left(\frac{x}{p}\right) + \left\{ \begin{array}{l} 0 \\ -\frac{1}{2}\left(\frac{a}{p}\right) \left(2 + \left(\frac{2}{p}\right)\right)h\left(-p\right) \\ p \equiv 3 \pmod{4} \right) \end{array} \right\}$$

where d is such that $0 \leq d \leq p - 1$ and $b^{s} - 4ac \equiv 4ad \pmod{p}$.

Proof. Define r by $b \equiv 2 ar \pmod{p}$, $0 \leq r \leq p-1$.

Then since
$$\left\{\frac{x}{p}\right\}$$
 is periodic with period p
$$\sum_{x=0}^{p-1} \left\{\frac{ax^2 + bx + c}{p}\right\} = \sum_{x=0}^{p-1} \left\{\frac{a(x+r)^2 + (c-ar^2)}{p}\right\}$$

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and the result follows on using the lemma.

In general, $\sum_{x=0}^{d-1} \left(\frac{x}{p}\right)$ cannot be given more simply. However when $p \equiv 1 \pmod{4}$ and d = 0, 1, 2, p' or p' + 1 its value is immediate.

In these special cases, setting $p'' = \frac{p+1}{2} = p' + 1$, we have immediately from the theorem :

Corollary : For $p \equiv 1 \pmod{4}$, p+a,

(i)
$$\sum_{x=0}^{p-1} \left\{ \frac{ax^{2}}{p} \right\} = p'$$

(11)
$$\sum_{x=0}^{p-1} \left\{ \frac{ax^{a}-1}{p} \right\} = p'$$

(iii)
$$\sum_{x=0}^{p-1} \left\{ \frac{ax^2-2}{p} \right\} = p' + \left(\frac{a}{p} \right)$$

(iv)
$$\sum_{x=0}^{p-1} \left\{ \frac{ax^2 - p'}{p} \right\} = p' + (-1)^{p'/2} + 1\left(\frac{a}{p}\right)$$

(v)
$$\sum_{x=0}^{p-1} \left\{ \frac{ax^{2} - p^{*}}{p} \right\} = p^{*}$$

From (iv) and (v) with a = 1 and $p \equiv 5 \pmod{8}$ I have the "reciprocal" relations

$$\sum_{x=0}^{p-1} \left\{ \frac{x^2 - p'}{p} \right\} = p^*, \ \sum_{x=0}^{p-1} \left\{ \frac{x^2 - p^*}{p} \right\} = p'$$

REFERENCES

- 1. I.M. Vinogradov, Elements of Number Theory (Dover) 1954 (See Ex. 2 (a) (a) P. 50.)
- 2. L.E. Dickson "History of the Theory of Numbers". Vol. 3. (Chelsea) 1952 (See . p 118).

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