Math. Gaz. 46 (1962), 315-316.

**3039. On the divisibility of $F_6$ by 274,177**

In 1640 the French Mathematician, Pierre Fermat (1601–65) asserted that the numbers defined by $F_n = 2^{2^n} + 1$ (so that $F_1 = 5$, $F_2 = 17$, $F_3 = 257$, $F_4 = 65{,}537$ etc.) are prime for all integral values of $n$. The first four are prime but, however, in 1732, the great Swiss Mathematician, Leonhard Euler (1707–83) found that

$$F_5 = 2^{2^5} + 1 = 2^{32} + 1 = 4{,}294{,}967{,}297 = 641 \,.\, 6{,}700{,}417$$

is composite.

Later, in 1880, Landry proved that

$$\begin{aligned} F_6 = 2^{2^6} + 1 = 2^{64} + 1 &= 18{,}446{,}744{,}073{,}709{,}551{,}617 \\ &= 274{,}177 \,.\, 67{,}280{,}421{,}310{,}721 \end{aligned}$$

was also composite. Here is a simple proof that 274,177 divides $F_6$ involving very little numerical calculation. We first show that

$$274{,}177 = 1071 \,.\, 2^8 + 1 = 516^2 + 89^2.$$

Then, working modulo 274,177, we prove a result we need later in the proof.

*Lemma.*

$$\begin{aligned} 89 \,.\, 15{,}409 &\equiv 516 \\ \therefore\ 89^2 \,.\, 15{,}409^2 &\equiv 516^2 \equiv -89^2 \\ \therefore\ 15{,}409^2 &\equiv -1. \end{aligned}$$

*Proof.* Now

$$2^{12}+1=(2^4)^3+1=(2^4+1)((2^4)^2-2^4+1)=17\,.\,241$$

and thus

$$\begin{aligned}2^{24}-1&=(2^3-1)(2^3+1)(2^6+1)(2^{12}+1)\\&=7\,.\,9\,.\,65\,.\,17\,.\,241\\&=(7\,.\,9\,.\,17)\,.\,(65\,.\,241)\\&=1071\,.\,15{,}665\\&=1071\,.\,2^8+1071\,.\,15{,}409\end{aligned}$$

$$\therefore\quad 2^{24}=1+1071\,.\,2^8+1071\,.\,15{,}409\equiv 1071\,.\,15{,}409$$

$$\therefore\quad 2^{48}\equiv 1071^2\,.\,15{,}409^2\equiv -1071^2\quad\text{(using lemma)}$$

$$\therefore\quad 2^{64}\equiv -1071^2\,.\,2^{16}\equiv -(274{,}177-1)^2\equiv -1$$

$$\therefore\ 2^{64}+1\equiv 0\pmod{274{,}177}$$

$$\therefore\ 274{,}177\mid F_6.$$

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