Chapter 9, Question 24

8. Let K be a quadratic field. Let $\alpha \in O_K$ be such that $|N(\alpha)| = ab$, where a and b are coprime positive integers. Prove that

$$< a, \alpha >< b, \alpha >=< \alpha > .$$

Solution. As K is a quadratic field, α has two conjugates, α and α'. As $\alpha \in O_K$ we have $\alpha' \in O_K$. Also

$$ab = |N(\alpha)| = |\alpha\alpha'|.$$

Thus

$$\alpha\alpha' = \pm ab.$$

Hence

$$\begin{aligned} < a, \alpha >< b, \alpha > &=< ab, \alpha a, \alpha b, \alpha^2 > \\ &=< \alpha\alpha', \alpha a, \alpha b, \alpha^2 > \\ &=< \alpha >< \alpha', a, b, \alpha > . \end{aligned}$$

Now a and b are coprime positive integers so there exist integers r and s such that

$$1 = ra + sb.$$

Hence

$$1 \in < \alpha', a, b, \alpha > .$$

Thus

$$< \alpha', a, b, \alpha >=< 1 >$$

and

$$< a, \alpha >< b, \alpha >=< \alpha > . \qquad \blacksquare$$