Chapter 6, Question 9

9. Let $K = \mathbb{Q}(\theta)$, where $\theta^3 - 4\theta + 2 = 0$. Let $\alpha = \theta + \theta^2 \in K$. Determine $D(\alpha)$.

Solution. The polynomial $x^3 - 4x + 2$ is 2-Eisenstein and so is irreducible. Thus $\mathrm{irr}_{\mathbb{Q}}(\theta) = x^3 - 4x + 2$. Hence $[K : \mathbb{Q}] = [\mathbb{Q}(\theta) : \mathbb{Q}] = \deg(x^3 - 4x + 2) = 3$. Thus K is a cubic field and the conjugates of θ with respect to K are the roots of $\theta, \theta', \theta''$ of $x^3 - 4x + 2$. Thus

$$\begin{aligned} D(\theta) = D(1, \theta, \theta^2) &= \mathrm{disc}(x^3 - 4x + 2) \\ &= -4(-4)^3 - 27(2)^2 = 256 - 108 = 148. \end{aligned}$$

Now let $\alpha = \theta + \theta^2$. As $\theta^3 - 4\theta + 2 = 0$ we have

$$\theta^3 = -2 + 4\theta, \ \theta^4 = -2\theta + 4\theta^2,$$

so that

$$\begin{aligned} \alpha^2 &= \left(\theta + \theta^2\right)^2 = \theta^2 + 2\theta^3 + \theta^4 \\ &= \theta^2 + 2(-2 + 4\theta) + (-2\theta + 4\theta^2) \\ &= -4 + 6\theta + 5\theta^2. \end{aligned}$$

Hence

$$\begin{aligned} D(\alpha) &= D(1, \alpha, \alpha^2) = D(1, \theta + \theta^2, -4 + 6\theta + 5\theta^2) \\ &= \begin{vmatrix} 1 & \theta + \theta^2 & -4 + 6\theta + 5\theta^2 \\ 1 & \theta' + \theta'^2 & -4 + 6\theta' + 5\theta'^2 \\ 1 & \theta'' + \theta''^2 & -4 + 6\theta'' + 5\theta''^2 \end{vmatrix}^2 \\ &= \begin{vmatrix} 1 & \theta & \theta^2 \\ 1 & \theta' & \theta'^2 \\ 1 & \theta'' & \theta''^2 \end{vmatrix}^2 \\ &= \begin{vmatrix} 1 & 0 & -4 \\ 0 & 1 & 6 \\ 0 & 1 & 5 \end{vmatrix}^2 \\ &= D(1, \theta, \theta^2) \begin{vmatrix} 1 & 6 \\ 1 & 5 \end{vmatrix} = D(\theta)(-1)^2 = 148. \end{aligned}$$

■

February 25, 2004