Chapter 6, Question 15

15. Prove that the discriminant D of the cubic polynomial $x^3+ax^2+bx+c \in \mathbb{Z}[x]$ is

$$D = a^2b^2 - 4b^3 - 4a^3c - 27c^2 + 18abc.$$

Deduce that $D \equiv 0$ or $1 \pmod 4$.

Solution. Let $x_1, x_2, x_3 \in \mathbb{C}$ be the three roots of $x^3 + ax^2 + bx + c$ so that

$$\begin{aligned} x_1 + x_2 + x_3 &= -a \\ x_1x_2 + x_2x_3 + x_3x_1 &= b, \\ x_1x_2x_3 &= -c. \end{aligned}$$

The discriminant of $x^3 + ax^2 + bx + c$ is

$$D = \{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)\}^2.$$

Now

$$\begin{aligned} &(x_1 - x_2)(x_1 - x_3)(x_2 - x_3) \\ &\qquad = (x_1^2x_2 + x_2^2x_3 + x_3^2x_1) - (x_1x_2^2 + x_2x_3^2 + x_3x_1^2) \\ &\qquad = A - B \text{ (say)} \end{aligned}$$

so that

$$D = (A - B)^2 = (A + B)^2 - 4AB.$$

First we compute

$$\begin{aligned} A + B &= x_1^2x_2 + x_1x_2^2 + x_2^2x_3 + x_2x_3^2 + x_3^2x_1 + x_3x_1^2 \\ &= x_1x_2(x_1 + x_2) + x_2x_3(x_2 + x_3) + x_1x_3(x_1 + x_3) \\ &= x_1x_2(-a - x_3) + x_2x_3(-a - x_1) + x_1x_3(-a - x_2) \\ &= -a(x_1x_2 + x_2x_3 + x_3x_1) - 3x_1x_2x_3 \\ &= -ab + 3c. \end{aligned}$$

Next we consider

$$\begin{aligned} AB &= (x_1^2x_2 + x_2^2x_3 + x_3^2x_1)(x_1x_2^2 + x_2x_3^2 + x_3x_1^2) \\ &= (x_1^3x_2^3 + x_2^3x_3^3 + x_3^3x_1^3) + (x_1^4x_2x_3 + x_1x_2^4x_3 + x_1x_2x_3^4) + 3x_1^2x_2^2x_3^2 \\ &= (x_1^3x_2^3 + x_2^3x_3^3 + x_3^3x_1^3) - c(x_1^3 + x_2^3 + x_3^3) + 3c^2. \end{aligned}$$

In order to determine $x_1^3 + x_2^3 + x_3^3$ and $x_1^3x_2^3 + x_2^3x_3^3 + x_3^3x_1^3$ we make use of the identity

$$y_1^3 + y_2^3 + y_3^3 = (y_1 + y_2 + y_3)^3 - 3(y_1 + y_2 + y_3)(y_1y_2 + y_2y_3 + y_3y_1) + 3y_1y_2y_3,$$

which is easily chacked. Taking $y_1 = x_1,\ y_2 = x_2,\ y_3 = x_3$, we obtain

$$x_1^3 + x_2^3 + x_3^3 = -a^3 + 3ab - 3c,$$

and taking $y_1 = x_1x_2,\ y_2 = x_2x_3,\ y_3 = x_3x_1$, we obtain, as $y_1 + y_2 + y_3 = x_1x_2 + x_2x_3 + x_3x_1 = b,\ y_1y_2 + y_2y_3 + y_3y_1 = x_1x_2x_3(x_1 + x_2 + x_3) = ac$ and $y_1y_2y_3 = (x_1x_2x_3)^2 = c^2$,

$$x_1^3x_2^3 + x_2^3x_3^3 + x_3^3x_1^3 = b^3 - 3abc + 3c^2.$$

Hence

$$\begin{aligned} AB &= (b^3 - 3abc + 3c^2) - c(-a^3 + 3ab - 3c) + 3c^2 \\ &= a^3c + b^3 - 6abc + 9c^2. \end{aligned}$$

Finally

$$\begin{aligned} D &= (-ab + 3c)^2 - 4(a^3c + b^3 - 6abc + 9c^2) \\ &= a^2b^2 - 4b^3 - 4a^3c - 27c^2 + 18abc. \end{aligned}$$

Clearly

$$D \equiv a^2b^2 + c^2 + 2abc \equiv (ab + c)^2 \equiv 0 \text{ or } 1 \pmod 4. \qquad \blacksquare$$

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