CHAPTER 5, QUESTION 12

12. Let $\theta \in \mathbb{C}$ be a root of $x^3 + 11x + 4 = 0$. Prove that $[\mathbb{Q}(\theta) : \mathbb{Q}] = 3$.

Solution. Let $f(x) = x^3 + 11x + 4 \in \mathbb{Z}[x]$. The only possible linear factors of $f(x)$ in $\mathbb{Z}[x]$ are $x - a$, where $a \mid 4$. However, none of these is a factor as

$$\begin{aligned}
f(-4) &= (-4)^3 + 11(-4) + 4 = -64 - 44 + 4 = -104 \neq 0, \\
f(-2) &= (-2)^3 + 11(-2) + 4 = -8 - 22 + 4 = -26 \neq 0, \\
f(-1) &= (-1)^3 + 11(-1) + 4 = -1 - 11 + 4 = -8 \neq 0, \\
f(1) &= 1^3 + 11(1) + 4 = 1 + 11 + 4 = 16 \neq 0, \\
f(2) &= 2^3 + 11(2) + 4 = 8 + 22 + 4 = 34 \neq 0, \\
f(4) &= 4^3 + 11(4) + 4 = 64 + 44 + 4 = 112 \neq 0.
\end{aligned}$$

Hence $f(x)$ is irreducible in $\mathbb{Z}[x]$. Thus the minimal polynomial of θ over $\mathbb{Q}$ is $f(x)$, so that $[\mathbb{Q}(\theta) : \mathbb{Q}] = \deg f(x) = 3$. ∎

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