CHAPTER 2, QUESTION 22

22. Prove that the subdomain $\mathbb{Z}+3\mathbb{Z}\sqrt{-2}$ of the Euclidean domain $\mathbb{Z}+\mathbb{Z}\sqrt{-2}$ is not Euclidean.

Solution. Let $D = \mathbb{Z} + 3\mathbb{Z}\sqrt{-2} = \mathbb{Z} + 3\mathbb{Z}\sqrt{-18}$. By Question 3 of Exercise 1 we see that $U(D) = \{\pm 1\}$. Hence $\tilde{D} = \{-1, 0, 1\}$. By Question 36 of Exercises 1, 2 and 3 are irreducible in D. Suppose that u is a universal side divisor in D. Then, u must divide one of the $2-1,\ 2-0,\ 2+1$, that is, one of $1,\ 2,\ 3$. But u being a universal side divisor is not a unit so $u \nmid 1$. Hence $u \mid 2$ or $u \mid 3$. Both 2 and 3 are irreducibles so $u = 2,\ -2,\ 3$ or -3. However none of these divides

$$\sqrt{-18}-1, \quad \sqrt{-18}, \quad \sqrt{-18}+1.$$

This is a contradiction. Hence D contains no universal side divisors. Thus, by Theorem 2.3.6, D is not Euclidean. ∎

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