CHAPTER 1, QUESTION 35

35. Let $r \in \mathbb{Z} \setminus \{-2, 0\}$. Prove that

$$D = \{a + b\theta + c\theta^2 \mid a, b, c \in \mathbb{Z}\},$$

where

$$\theta^3 + r\theta + 1 = 0,$$

is an integral domain. Prove that $\theta \in U(D)$.

Solution. As $r \neq -2, 0$ the cubic polynomial $x^3 + rx + 1$ is irreducible in $\mathbb{Z}[x]$ so that D is a cubic domain. Let $\alpha = -r - \theta^2$. Clearly $\alpha \in D$. Moreover

$$\theta\alpha = \theta(-r - \theta^2) = -r\theta - \theta^3 = 1.$$

Thus $\theta \in D$ is a unit. Hence $\theta \in U(D)$. ∎

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