CHAPTER 1, QUESTION 3

3. Let m be an integer with $m < -1$. Prove that

$$U(\mathbb{Z} + \mathbb{Z}\sqrt{m}) = \{\pm 1\}.$$

Solution. Let $\alpha \in U(\mathbb{Z} + \mathbb{Z}\sqrt{m})$, where m is an integer with $m < -1$. Then there exists $\beta \in \mathbb{Z} + \mathbb{Z}\sqrt{m}$ such that

$$\alpha\beta = 1.$$

As $\alpha, \beta \in \mathbb{Z} + \mathbb{Z}\sqrt{m}$ there exist $a, b, c, d \in \mathbb{Z}$ such that $\alpha = a + b\sqrt{m},\ \beta = c + d\sqrt{m}$. Thus

$$(a + b\sqrt{m})(c + d\sqrt{m}) = 1.$$

Hence

$$(ac + bdm) + (ad + bc)\sqrt{m} = 1.$$

As $m < -1,\ \sqrt{m} \in \mathbb{C} \setminus \mathbb{R}$ so that

$$ac + bdm = 1,\ ad + bc = 0.$$

Thus

$$(a^2 - mb^2)(c^2 - md^2) = (ac + bdm)^2 - m(ad + bc)^2 = 1.$$

As $m < -1,\ a^2 - mb^2$ is a positive integer dividing 1. Hence

$$a^2 - mb^2 = 1.$$

If $b \neq 0$ then, as $m < -1$, we have

$$1 = a^2 - mb^2 \geq -mb^2 > b^2 \geq 1,$$

a contradiction. Hence $b = 0$ and $a = \pm 1$. Thus $\alpha = \pm 1$ proving

$$U(\mathbb{Z} + \mathbb{Z}\sqrt{m}) \subseteq \{-1, 1\}.$$

Clearly $\pm 1 \in U(\mathbb{Z} + \mathbb{Z}\sqrt{m})$ so that

$$\{-1, 1\} \subseteq U(\mathbb{Z} + \mathbb{Z}\sqrt{m}).$$

This completes the proof that

$$U(\mathbb{Z} + \mathbb{Z}\sqrt{m}) = \{\pm 1\}$$

for $m < -1$. $\blacksquare$

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