CHAPTER 1, QUESTION 22

22. Let D be an integral domain. Let $a, b, c \in D$ be such that $< a, c >= D$. Prove that $< a, bc >=< a, b >$.

Solution. As $< a, c >= D$ and $1 \in D$ there exist $r, s \in D$ such that

$$ra + sc = 1.$$

Let $\alpha \in< a, b >$. Then there exist $k, l \in D$ such that

$$\alpha = ka + lb.$$

Hence

$$\begin{aligned} \alpha &= ka + lb(ra + sc) \\ &= (k + lbr)a + (ls)bc \\ &\in< a, bc > . \end{aligned}$$

This proves that

$$< a, b >\subseteq< a, bc > .$$

Now suppose that $\beta \in< a, bc >$. Then there exist $u, v \in D$ such that

$$\beta = ua + vbc.$$

Set $w = vc \in D$. Then $\beta = ua + wb \in< a, b >$. This proves that

$$< a, bc >\subseteq< a, b > .$$

The two inclusions give $< a, bc >=< a, b >$. ∎

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