CHAPTER 1, QUESTION 14

14. Let $A$ and $B$ be ideals of an integral domain $D$. Show that $(A \cap B)(A + B) = AB$.

Solution. Let $K = A \cap B$ and $L = A + B$. $K$ and $L$ are ideals of $D$. We wish to prove that

$$KL \subseteq AB.$$

Let $\alpha \in KL$. Then

$$\alpha = k_1 l_1 + \cdots + k_m l_m$$

for some $k_1, \ldots, k_m \in K$ and $l_1, \ldots, l_m \in L$. As $L = A + B$ we have

$$l_i = a_i + b_i, \ i = 1, \ldots, m,$$

where $a_i \in A, \ b_i \in B$. Hence

$$\begin{aligned} \alpha &= k_1(a_1 + b_1) + \cdots + k_m(a_m + b_m) \\ &= a_1 k_1 + \cdots + a_m k_m + k_1 b_1 + \cdots k_m b_m. \end{aligned}$$

As $K \subseteq B$ we see that

$$k_1, \ldots, k_m \in B$$

so that

$$a_1 k_1 + \cdots + a_m k_m \in AB.$$

As $K \subseteq A$, we see that

$$k_1, \ldots, k_m \in A$$

so that

$$k_1 b_1 + \cdots + k_m b_m \in AB.$$

As $AB$ is an ideal

$$a_1 k_1 + \cdots + a_m k_m + k_1 b_1 + \cdots + k_m b_m \in AB,$$

that is

$$\alpha \in AB,$$

so that

$$KL \subseteq AB$$

as required.

We have shown above that

$$(A \cap B)(A + B) \subseteq AB. \tag{1}$$

We now show that equality does not always hold in (1).

Let $D = \mathbb{Z}[x]$, $A =< 2 >$ and $B =< x >$ so that

$$AB =< 2x >, \ A \cap B =< 2x >, \ A + B =< 2, x > .$$

Then

$$(A \cap B)(A + B) =< 2x >< 2, x >=< 4x, 2x^2 > .$$

Clearly $2x \notin < 4x, 2x^2 >$ so that $< 4x, 2x^2 > \neq < 2x >$. Hence $(A \cap B)(A + B) \neq AB$.

■

Remark. From Question 13 we have

$$AB \subseteq A \cap B$$

and from Question 14 that

$$(A \cap B)(A + B) \subseteq AB. \tag{2}$$

Hence

$$(A \cap B)(A + B) \subseteq AB \subseteq A \cap B. \tag{3}$$

Thus if $A + B =< 1 >$ we deduce that

$$A \cap B = AB.$$

June 19, 2004