On inverse permutation polynomials

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Abstract

We give an explicit formula of the inverse polynomial of a permutation polynomial of the form $x^r f(x^s)$ over a finite field $\mathbb{F}_q$ where $s \mid q-1$. This generalizes results in [6] where $s=1$ or $f = g^{\frac{q-1}{s}}$ were considered respectively. We also apply our result to several interesting classes of permutation polynomials.

Key words: permutation polynomials, inverse polynomials, generalized Lucas sequence, finite fields
MSC: 11T06, 11B39

1 Introduction

Let p be prime, $q = p^m$, and $\mathbb{F}_q$ be a finite field of order q. Let $P(x)$ be a permutation polynomial (PP) over $\mathbb{F}_q$ and $Q(x)$ be the compositional inverse polynomial of $P(x)$. By the modulo reduction $x^q - x$, we only need to consider polynomials of degree less than or equal to $q-1$. Because a permutation polynomial can not have degree $q-1$, we let $P(x) = a_0 + a_1 x + \cdots + a_{q-2}x^{q-2}$ be a permutation polynomial of $\mathbb{F}_q$ and $Q(x) = b_0 + b_1 x + \cdots + b_{q-2}x^{q-2}$ be the inverse polynomial of $P(x)$ modulo $x^q - x$. In [5], G. L. Mullen posed the problem of computing the coefficients of the inverse polynomial of a permutation polynomial efficiently (Problem 10). Recently Muratović-Ribić [6] characterized all the coefficients of the inverse polynomial of a permutation polynomial of the form $x^r f(x^s)^{(q-1)/s}$ as follows:

Theorem 1.1 (Muratović-Ribić) *Let $P(x) = x^r f(x^s)^{\frac{q-1}{s}} \in \mathbb{F}_q[x]$ where $r \geq 1$ is an integer with $\gcd(r, q-1) = 1$, s is a divisor of $q-1$ and $f(x) \in \mathbb{F}_q[x]$ is a polynomial without roots in $\mathbb{F}_q$. Denote by $Q(x) = b_0 + b_1 x + \cdots + b_{q-2}x^{q-2}$ the inverse of permutation*

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polynomial $P(x)$ modulo $x^q - x$. Let k_0 be the least positive integer for which there exists a positive integer l_0 such that $l_0 s = k_0 r + 1$ and

$$f(x^s)^{\frac{q-1}{s}k_0} \equiv \sum_{i=0}^{(q-1)/s} d_i x^{is} \pmod{x^q - x}.$$

Then $b_n \neq 0$ only if $s \mid rn - 1$. Moreover, if $b_n \neq 0$, then the following holds:
(i) If $rn \not\equiv 1 \pmod{q-1}$ and $i \equiv \frac{rn-1}{s} \pmod{\frac{q-1}{s}}$ then $b_n = d_i$.
(ii) If $rn \equiv 1 \pmod{q-1}$ then $b_n = d_0 + d_{(q-1)/s}$.

The method used in the proof of Theorem 1.1 is based on Equation (3) in [6] which applies to more general polynomial $P(x)$, for example, $P(x) = x^r f(x^s)$ where $s = 1$.

It is well-known that any nonconstant polynomial $h(x) \in \mathbb{F}_q[x]$ can be written as $ax^r f(x^s) + b$ where $a \neq 0$ and $s \mid q-1$. To find the inverse of $h(x)$, it is enough to find the inverse of permutation polynomial $x^r f(x^s)$. We refer to [4] or [8] for some general characterization of permutation polynomials $P(x) = x^r f(x^s)$. For $s = 1$, an explicit formula of the inverse of permutation polynomial $x^r f(x)$ is obtained directly from Equation (3) in [6]. In this paper, we use the similar method as in [6] to give an explicit formula of the inverse polynomial of a permutation polynomial of the form $x^r f(x^s)$ over a finite field $\mathbb{F}_q$ for any $s \mid q-1$ (Theorem 2.1). We also apply Theorem 2.1 to several interesting classes of permutation polynomials considered in [4]. These results (Corollaries 2.3, 2.4) are presented in Section 2. Finally we explore the connection (Theorem 3.1) between inverse polynomials of permutation binomials of the form $x^r(x^{es}+1)$ over $\mathbb{F}_q$ and so-called generalized Lucas sequences over $\mathbb{F}_p$. Some examples of inverse polynomials of permutation binomials are also provided in Section 3.

2 General results

Let us assume that $P(x) = x^r f(x^s)$ is a permutation polynomial of $\mathbb{F}_q$. It is well known that if $P(x) = x^r f(x^s)$ is a permutation polynomial of $\mathbb{F}_q$ then we must have $(r, s) = 1$. Hence the inverse of r modulo s exists and we denote it by $\bar{r} = r^{-1} \bmod s$. The notation $a = b \bmod c$ means that a is an integer such that $0 \leq a < c$ and $a \equiv b \pmod{c}$. We will use this notation and the fact $\bar{r} = r^{-1} \bmod s$ frequently later on.

First we show that the inverse polynomial $Q(x)$ of $P(x) = x^r f(x^s)$ has at most $\ell := \frac{q-1}{s}$ nonzero coefficients and give the explicit formula to compute these coefficients. We assume that $\ell \geq 2$ in this paper since $\ell = 1$ is the trivial case.

Theorem 2.1 *Let $P(x) = x^r f(x^s) \in \mathbb{F}_q[x]$ be a permutation polynomial of $\mathbb{F}_q$ where $r \geq 1$, $s = \frac{q-1}{\ell}$, $\ell \geq 2$ is a divisor of $q-1$. Denote by $Q(x) = b_0 + b_1 x + \cdots + b_{q-2} x^{q-2}$ the inverse polynomial of $P(x)$ modulo $x^q - x$. Then the following holds.*

(i) If $b_n \neq 0$, then $s \mid (rn-1)$. In particular, there are at most ℓ such nonzero b_n's such

that $0 \le n \le q-2$ *and* $n \equiv r^{-1} \pmod{s}$*. That is,* $n = is + \bar{r}$ *where* $i = 0, \cdots, \ell-1$ *and* $\bar{r} = r^{-1} \bmod s$*.*

(ii) Let $\bar{a} \equiv \frac{r\bar{r}-1}{s} \pmod{\ell}$*. Then*

$$b_{is+\bar{r}} = \frac{1}{\ell}\sum_{t=0}^{\ell-1} \zeta^{-(ir+\bar{a})t} f(\zeta^t)^{q-1-\bar{r}-is}, \quad i = 0, \cdots, \ell-1,$$

where ζ *is a primitive* ℓ*-th root of unity.*

(iii) For each $i = 0, \cdots, \ell-1$*, let* $f(x^s)^{q-1-\bar{r}-is} \equiv \sum_{j=0}^{\ell} d_{i,j} x^{js} \pmod{x^q - x}$ *and* $m_i = ir + \bar{a} \bmod \ell$*. Then* $b_{is+\bar{r}} = d_{i,m_i}$ *if* $m_i \neq 0$ *and* $b_{is+\bar{r}} = d_{i,0} + d_{i,\ell}$ *if* $m_i = 0$*.*

Proof. By Equation (3) in [6],

$$b_n = -\sum_{x \in \mathbb{F}_q} xP(x)^{q-1-n} = -\sum_{x \in \mathbb{F}_q} x \sum_{i=0}^{q-1} c_i x^i = c_{q-2},$$

where $P(x)^{q-1-n} \pmod{x^q - x} = c_0 + c_1 x + \cdots + c_{q-1}x^{q-1}$. If b_n is nonzero, then the coefficient of x^{q-2} in the expansion of $P(x)^{q-1-n}$ is nonzero. Hence there exists some j such that $js + r(q-1) - rn \equiv q-2 \pmod{q-1}$ and thus $js \equiv rn - 1 \pmod{q-1}$. Therefore, $s \mid (rn-1)$. That is, $rn \equiv 1 \pmod{s}$. Because $(r, s) = 1$, we have $n \equiv r^{-1} \pmod{s}$. Therefore there are at most ℓ nonzero coefficients in the inverse polynomial $Q(x)$ corresponding to $n \equiv r^{-1} \pmod{s}$. Hence $n = is + \bar{r}$ for $i = 0, \cdots, \ell-1$ where $\bar{r} = r^{-1} \bmod s$. It is therefore straightforward to obtain $b_{is+\bar{r}} = -\sum_{s \in \mathbb{F}_q} xP(x)^{q-1-is-\bar{r}} = \frac{1}{\ell}\sum_{t=0}^{\ell-1} \zeta^{-(ir+\bar{a})t} f(\zeta^t)^{q-1-\bar{r}-is}$.

Finally, $q-1 = \ell s$ implies that $-s$ and $\frac{1}{\ell}$ are the same in $\mathbb{F}_q$. Since $m_i = ir + \bar{a} \bmod \ell$, we have

$$\begin{aligned}\frac{1}{\ell}\sum_{t=0}^{\ell-1} \zeta^{-(ir+\bar{a})t} f(\zeta^t)^{q-1-\bar{r}-is} &= -s\sum_{t=0}^{\ell-1} \zeta^{-(ir+\bar{a})t} f(\zeta^t)^{q-1-\bar{r}-is} \\ &= -\sum_{x \in \mathbb{F}_q} x^{q-1-m_i s} f(x^s)^{q-1-\bar{r}-is}.\end{aligned}$$

However, the last term is equal to d_{i,m_i} if $m_i \neq 0$ and is equal to $d_{i,0} + d_{i,\ell}$ otherwise. □

Remark: For positive integers n, ℓ, a, the lacunary sum for the coefficient $C(n, j, k)$ of x^j in the polynomial expansion of $f(x)^n = (f_0 + f_1 x + f_2 x^2 + ... + f_k x^k)^n$ is defined as

$$S(n, \ell, a, k+1) = \sum_{\substack{j=0 \\ j \equiv a \pmod{\ell}}}^{nk} C(n, j, k),$$

where

$$C(n,j,k)=\sum_{\substack{n_0+n_1+\cdots+n_k=n\\ n_1+2n_2+\cdots+kn_k=j}}\frac{n!}{n_0!\,n_1!\cdots n_k!}f_0^{n_0}f_1^{n_1}\cdots f_k^{n_k}.$$

Using

$$\sum_{\substack{j=0\\ j\equiv a(\mathrm{mod}\ \ell)}}^{nk} C(n,j,k)=\frac{1}{\ell}\sum_{t=0}^{\ell-1}\zeta^{-at}\sum_{j=0}^{nk}C(n,j,k)\zeta^{jt}=\frac{1}{\ell}\sum_{t=0}^{\ell-1}\zeta^{-at}f(\zeta^t)^n, \tag{1}$$

we obtain that

$$S(n,\ell,a,k+1)=\frac{1}{\ell}\sum_{t=0}^{\ell-1}\zeta^{-at}f(\zeta^t)^n. \tag{2}$$

Hence (ii) of Theorem 2.1 can also be written as

$$b_{is+\bar r}=S(q-1-\bar r-is,\ell,ir+\bar a,k+1),\quad i=0,\cdots,\ell-1, \tag{3}$$

From the above theorem, we need to compute ℓ different powers of $f(x^s)$ in order to find all the coefficients of the inverse polynomial of $P(x)$. We note that it is not efficient to find all the coefficients of the inverse polynomial if $s=1$. However, if s is big (i.e., ℓ is small), it is quite efficient to compute the inverse polynomial by using the above theorem. For example, for odd q, it is well known that $P(x)=x^r f(x^{(q-1)/2})$ is a permutation polynomial of $\mathbb{F}_q$ if and only if $(r,(q-1)/2)=1$ and $(f(-1)f(1))^{\frac{q-1}{2}}=(-1)^{r+1}$. The next result gives the explicit format of the inverse polynomial of such permutation polynomial by applying Theorem 2.1.

Corollary 2.2 *For odd q and $s=\frac{q-1}{2}$, the inverse polynomial $Q(x)$ of the permutation polynomial $P(x)=x^r f(x^s)$ is given by $b_{\bar r}x^{\bar r}+b_{s+\bar r}x^{s+\bar r}$ with $b_{\bar r}=\frac{1}{2}(f(1)^{q-1-\bar r}+(-1)^{\bar a}f(-1)^{q-1-\bar r})$ and $b_{s+\bar r}=\frac{1}{2}(f(1)^{s-\bar r}+(-1)^{\bar a'}f(-1)^{s-\bar r})$, where $\bar r=r^{-1} \bmod s$, $\bar a\equiv\frac{r\bar r-1}{s} \pmod 2$, $\bar a'\equiv\bar a+r \pmod 2$.*

Next we show in certain cases, we can also simplify this process by computing only one fixed power of each $f(x^s)$ even for large ℓ. The following theorem is one of such examples which also generalizes Theorem 1.1. Indeed, if $f(x)=g(x)^\ell$ then $f(x)^s=1$.

Corollary 2.3 *Let $q-1=\ell s$ and $P(x)=x^r f(x^s)\in\mathbb{F}_q[x]$ be a permutation polynomial of $\mathbb{F}_q$ where $r\geq 1$ and $s=\frac{q-1}{\ell}$. Denote by $Q(x)=b_0+b_1x+\cdots+b_{q-2}x^{q-2}$ its inverse polynomial modulo x^q-x. Assume that $f(\zeta^t)^s=1$ for a primitive ℓ-th root of unity ζ and any $t=0,\cdots,\ell-1$. Let $\bar r=r^{-1} \bmod s$ and $\bar a\equiv\frac{r\bar r-1}{s} \pmod{\ell}$. Then, for all possible nonzero coefficients b_n corresponding to $n=is+\bar r$ where $i=0,\cdots,\ell-1$, we have*

$$b_{is+\bar r}=\frac{1}{\ell}\sum_{t=0}^{\ell-1}\zeta^{-(ir+\bar a)t}f(\zeta^t)^{q-1-\bar r}.$$

In particular, assume $f(x^s)^{q-1-\bar r}\equiv\sum_{j=0}^{\ell}d_jx^{js} \pmod{x^q-x}$ and $m_i=ir+\bar a \bmod \ell$. Then $b_n=d_{m_i}$ if $m_i\neq 0$ and $b_n=d_0+d_\ell$ if $m_i=0$.

Proof. The first part follows immediately from Theorem 2.1 and $f(\zeta^t)^s = 1$. Because $q-1 = \ell s$, $-s$ and $\frac{1}{\ell}$ are the same in $\mathbb{F}_q$. Hence $\frac{1}{\ell}\sum_{t=0}^{\ell-1}\zeta^{-(ir+\bar{a})t}f(\zeta^t)^{q-1-\bar{r}} = -s\sum_{t=0}^{\ell-1}\zeta^{-(ir+\bar{a})t}f(\zeta^t)^{q-1-\bar{r}} = -\sum_{x\in\mathbb{F}_q}x^{q-1-(ir+\bar{a})s}f(x^s)^{q-1-\bar{r}}$. However, the last term is equal to d_{m_i} if $m_i \neq 0$ and is equal to $d_0 + d_\ell$ otherwise. Hence the proof is complete. □

By using a similar proof, we obtain

Corollary 2.4 *Let $q-1 = \ell s$ and $P(x) = x^r f(x^s) \in \mathbb{F}_q[x]$ be a permutation polynomial of $\mathbb{F}_q$ where $r \geq 1$ and $s = \frac{q-1}{\ell}$. Denote by $Q(x) = b_0 + b_1 x + \cdots + b_{q-2}x^{q-2}$ its inverse polynomial modulo $x^q - x$. Let $\bar{r} = r^{-1} \bmod s$ and $\bar{a} \equiv \frac{r\bar{r}-1}{s} \pmod{\ell}$. Assume that $f(\zeta^t)^s = \zeta^{kt}$ for a primitive ℓ-th root of unity ζ and any $t = 0, \cdots, \ell-1$. Then, for all possible nonzero coefficients b_n corresponding to $n = is + \bar{r}$ where $i = 0, \cdots, \ell-1$, we have*

$$b_{is+\bar{r}} = \frac{1}{\ell}\sum_{t=0}^{\ell-1}\zeta^{-(ir+\bar{a}+ik)t}f(\zeta^t)^{q-1-\bar{r}}.$$

In particular, assume $f(x^s)^{q-1-\bar{r}} \equiv \sum_{j=0}^{\ell} d_j x^{js} \pmod{x^q - x}$ and $m_i = ir + \bar{a} + ik \bmod \ell$. Then $b_n = d_{m_i}$ if $m_i \neq 0$ and $b_n = d_0 + d_\ell$ if $m_i = 0$.

We refer the readers to [4] for several interesting classes of permutation polynomials which satisfy the assumptions of Corollary 2.3 and Corollary 2.4.

3 Binomials and sequences

In this section, we consider the inverse polynomial of a permutation binomial $f(x) = x^r(x^{es}+1)$ over $\mathbb{F}_q$ where $q = p^m$, $q-1 = \ell s$ for some positive integers ℓ, s and $(e, \ell) = 1$. We note that the characterization of permutation polynomials of the form $x^r(x^{es}+1)$ have been studied by Akbary and the author in [2], [3] and [9]. In particular, if $f(x) = x^r(x^{es}+1)$ is a permutation polynomial over $\mathbb{F}_q$ then p must be odd. Otherwise, $P(0) = P(1) = 0$. Since $\ell \mid q-1$, let $\zeta \in \mathbb{F}_q$ be a primitive ℓ-th root of unity. Moreover, we must have $\zeta^{ei} \neq -1$ for $i = 0, \cdots, \ell-1$. Hence ℓ must be odd and then s must be even. So we can assume that $\ell \geq 3$ as $\ell = 1$ is trivial. Because both p and ℓ are odd, there exists $\eta \in \mathbb{F}_q$ such that $\eta^2 = \zeta$. Hence η is a primitive 2ℓ-th root of unity in $\mathbb{F}_q$.

We define the sequence $\{a_n\}_{n=0}^{\infty}$ by

$$a_n = \sum_{t=1}^{\frac{\ell-1}{2}}\left((-1)^{t+1}(\eta^t + \eta^{-t})\right)^n = \sum_{\substack{t=1\\ t\ odd}}^{\ell-1}\left(\eta^t + \eta^{-t}\right)^n.$$

The sequence $\{a_n\}_{n=0}^{\infty}$ is called *generalized Lucas sequence of order* $\frac{\ell-1}{2}$ because $\{a_n\}_{n=0}^{\infty} = \{L_n\}_{n=0}^{\infty}$ when $\ell = 5$, where the sequence $\{L_n\}_{n=0}^{\infty}$ is the so-called Lucas sequence satisfying the recurrence relation $L_{n+2} - L_{n+1} - L_n = 0$ and $L_0 = 2$ and $L_1 = 1$.

For any integer $n \geq 1$, we recall that the Dickson polynomial of the first kind $D_n(x) \in \mathbb{F}_q[x]$ of degree n is defined by

$$D_n(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{n}{n-i} \binom{n-i}{i} (-1)^i x^{n-2i}.$$

Similarly, the Dickson polynomial of the second kind $E_n(x) \in \mathbb{F}_q[x]$ of degree n is defined by

$$E_n(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} (-1)^i x^{n-2i}.$$

We consider the Dickson polynomial $E_{\ell-1}(x)$ of the second kind with degree $\ell - 1$. It is well known that $\eta^t + \eta^{-t}$ with $1 \leq t \leq \ell - 1$ are all the roots of $E_{\ell-1}(x)$ where η is a primitive 2ℓ-th root of unity. Let

$$E_{\ell-1}^{odd}(x) = \prod_{\substack{t=1 \\ odd\ t}}^{\ell-1} (x - (\eta^t + \eta^{-t})).$$

Then the characteristic polynomial of the sequence $\{a_n\}_{n=0}^{\infty}$ is $E_{\ell-1}^{odd}(x)$ and $\{a_n\}_{n=0}^{\infty}$ is a sequence over the prime field $\mathbb{F}_p$.

Now we prove the following result which gives the explicit format of the inverse polynomials of permutation binomials of the form $x^r(x^{e(q-1)/\ell} + 1)$ in terms of generalized Lucas sequence of order $\frac{\ell-1}{2}$.

Theorem 3.1 *Let p be odd prime and $q = p^m$. Assume that ℓ, s, r, e are positive integers such that $\ell \geq 3$ is odd, $q - 1 = \ell s$, and $(e, \ell) = 1$. If $P(x) = x^r(x^{es} + 1)$ is a permutation polynomial of $\mathbb{F}_q$ and $Q(x) = b_0 + b_1 x + \cdots + b_{q-2} x^{q-2}$ is the inverse polynomial of $P(x)$ modulo $x^q - x$, then the following holds.*

(i) If $b_n \neq 0$, then $n \equiv r^{-1} \pmod{s}$. Hence $Q(x)$ has at most ℓ nonzero coefficients b_n corresponding to $n = is + \bar{r}$ where $\bar{r} = r^{-1} \bmod s$ and $i = 0, \cdots, \ell - 1$.

(ii)

$$b_n = \frac{1}{\ell}\left(2^{q-1-n} + \sum_{i=0}^{\lfloor u_n/2 \rfloor} t_i^{(u_n)} a_{q-1-n+u_n-2i}\right), \tag{4}$$

where $\bar{n} = \frac{rn-1}{s} \bmod \ell$, $u_n = 2\bar{n}e^{\phi(\ell)-1} + n \bmod 2\ell$, $t_i^{(u_n)} = \frac{u_n}{u_n - i}\binom{u_n - i}{i}(-1)^i$, and $\{a_n\}_{n=0}^{\infty}$ is the generalized Lucas sequence of order $\frac{\ell-1}{2}$.

Proof. By Theorem 2.1, $Q(x)$ has at most ℓ nonzero coefficients b_n with $n \equiv r^{-1} \pmod{s}$ and $1 \leq n \leq q - 2$. Then $n = is + \bar{r}$ where $\bar{r} = r^{-1} \bmod s$ and $i = 0, \cdots, \ell - 1$. Moreover, $\bar{n} \equiv \frac{rn-1}{s} \equiv ir + \bar{a} \pmod{\ell}$ where $\bar{a} \equiv \frac{r\bar{r}-1}{s} \pmod{\ell}$.

Let $\xi = \zeta^e$. Since $(e, \ell) = 1$, ξ is also a primitive ℓ-th root of unity. Moreover, because

$2\ell \mid q-1$, then there exists $\eta \in \mathbb{F}_q$ such that $\eta^2 = \xi$. Because ζ^{-1} is also a primitive ℓ-th root of unity, by Theorem 2.1, we obtain

$$\begin{aligned}
b_n &= \frac{1}{\ell}\sum_{t=0}^{\ell-1} \zeta^{\bar{n}t} f(\zeta^{-t})^{q-1-n} \\
&= \frac{1}{\ell}\sum_{t=0}^{\ell-1} \zeta^{\bar{n}t} (\zeta^{-et}+1)^{q-1-n} \\
&= \frac{1}{\ell}\sum_{t=0}^{\ell-1} \xi^{\bar{n}e^{\phi(\ell)-1}t} (\xi^{-t}+1)^{q-1-n} \\
&= \frac{1}{\ell}(2^{q-1-n} + \sum_{t=1}^{\ell-1} \eta^{2\bar{n}e^{\phi(\ell)-1}t-(q-1-n)t}(\eta^{-t}+\eta^{t})^{q-1-n}) \\
&= \frac{1}{\ell}(2^{q-1-n} + \sum_{t=1}^{\frac{\ell-1}{2}} (\eta^{(2\bar{n}e^{\phi(\ell)-1}+n)t} + \eta^{-(2\bar{n}e^{\phi(\ell)-1}+n)t})(\eta^{-t}+\eta^{t})^{q-1-n}),
\end{aligned}$$

where the last identity holds because q, n are odd and $\eta^\ell = -1$. Hence the result follows from the definition of $\{a_n\}_{n=0}^{\infty}$ and the fact

$$\eta^{u_n t} + \eta^{-u_n t} = D_{u_n}(\eta^t + \eta^{-t}) = \sum_{i=0}^{\lfloor u_n/2 \rfloor} \frac{u_n}{u_n - i}\binom{u_n - i}{i}(-1)^i(\eta^t+\eta^{-t})^{u_n-2i}.$$

This completes the proof. □

We note that the equation (4) can also be written as

$$b_{q-1-n} = \frac{1}{\ell}(2^n + \sum_{j=0}^{u_n} c_j^{(u_n)} a_{n+j}), \tag{5}$$

where $c_j^{(u_n)}$ is the coefficient of x^j in the expansion of the Dickson polynomial of the first kind $D_{u_n}(x)$ of degree $u_n = 2\hat{n}e^{\phi(\ell)-1} + (q-1-n) \pmod{2\ell}$ and $\hat{n} = \frac{(q-1-n)r-1}{s} \pmod{\ell}$. Moreover, all the coefficients of the inverse polynomial $Q(x)$ in Theorem 3.1 are in $\mathbb{F}_p$. Because the coefficients $t_i^{(u_n)}$ and the general term of generalized Lucas sequence $\{a_n\}_{n=0}^{\infty}$ over $\mathbb{F}_p$ are quite easy to find, one can generate many examples of inverse polynomials by applying Theorem 3.1. For example, if $\ell = 3$ and $s = (q-1)/3$, then $\{a_n\}_{n=0}^{\infty}$ is a constant sequence $1, 1, \cdots$. Hence $b_n = \frac{1}{3}(2^{-\bar{r}} + D_{u_n}(1))$ because $P(x) = x^r(x^{es}+1)$ is a permutation polynomial over $\mathbb{F}_q$ if and only if $(r, s) = 1$, $2^s \equiv 1 \pmod{p}$, and $(2r+es, \ell) = 1$. In the case $\ell = 5$ and $s = (q-1)/5$, the corresponding sequence $\{a_n\}_{n=0}^{\infty}$ is the Lucas sequence. In this case, $P(x) = x^r(x^{es}+1)$ is a permutation polynomial over $\mathbb{F}_q$ if and only if $(r, s) = 1$, $2^s \equiv 1 \pmod{p}$, $(2r+es, \ell) = 1$, $a_s = 2$. In particular, $\{a_n\}_{n=0}^{\infty}$ is periodic with a period s. Hence we can use s-periodicity of $\{a_n\}_{n=0}^{\infty}$ and $2^s \equiv 1 \pmod{p}$ to simplify the computation of equation (4) or equation (5). We observe that explicit formulas of inverse polynomials of permutation binomials for the cases $\ell = 3, 5$ have also been obtained recently by Muratović-Ribić in [7] without using sequences. The formulas in [7] are similar to Equation (3) for $\ell = 3, 5$. When $\ell \geq 7$, generalized Lucas sequences were introduced so that we can evaluate the lacunary sums. Here we give some examples of inverse polynomials of permutation binomials with $\ell \geq 7$.

Permutation binomials $x^r(x^{\frac{e(q-1)}{7}} + 1)$ and inverse polynomials over $\mathbb{F}_{13^2}$

PP	Inverse of PP
$x + x^{25}$	$7x + 7x^{25} + 6x^{49} + 7x^{73} + 6x^{97} + 7x^{121} + 6x^{145}$
$x^5 + x^{29}$	$2x^5 + 9x^{29} + 7x^{53} + 8x^{77} + 8x^{101} + 7x^{125} + 9x^{149}$
$x^7 + x^{31}$	$5x^7 + 5x^{55} + 10x^{79} + x^{103} + x^{127} + 10x^{151}$
$x^{11} + x^{35}$	$x^{59} + x^{131}$
$x^{13} + x^{37}$	$7x^{13} + 6x^{37} + 7x^{61} + 7x^{85} + 6x^{109} + 6x^{133} + 7x^{157}$
$x^{17} + x^{41}$	$9x^{17} + 9x^{41} + 8x^{65} + 7x^{89} + 2x^{113} + 7x^{137} + 8x^{161}$
$x^{19} + x^{43}$	$10x^{43} + x^{67} + 5x^{91} + 5x^{115} + x^{139} + 10x^{163}$
...	...

Permutation binomials $x^r(x^{\frac{e(q-1)}{9}} + 1)$ and inverse polynomials over $\mathbb{F}_{17^2}$

PP	Inverse of PP
$x + x^{33}$	$9x + 9x^{33} + 8x^{65} + 9x^{97} + 8x^{129} + 9x^{161} + 8x^{193} + 9x^{225} + 8x^{257}$
$x^3 + x^{35}$	$x^{11} + 5x^{43} + 10x^{75} + 10x^{107} + 5x^{139} + x^{171}$
$x^7 + x^{39}$	$16x^{23} + 9x^{55} + 7x^{87} + 2x^{119} + 7x^{151} + 9x^{183} + 16x^{215} + 2x^{247} + 2x^{279}$
$x^9 + x^{41}$	$4x^{25} + x^{57} + 7x^{89} + 7x^{153} + x^{185} + 4x^{217} + x^{249} + x^{281}$
$x^{13} + x^{45}$	$5x^5 + 12x^{37} + 3x^{69} + 7x^{101} + 5x^{133} + 5x^{165} + 7x^{197} + 3x^{229} + 12x^{261}$
$x^{15} + x^{47}$	$x^{47} + x^{111}$
$x^{19} + x^{51}$	$x^{27} + 5x^{59} + 10x^{91} + 10x^{123} + 5x^{155} + x^{187}$
...	...

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