HOMEWORK No 7 (March 9, 2005)

Exercise 1. *Construct a polynomial* $g(x) \in \mathbf{Z_5}[x]$ *of degree* ≤ 4 *such that*

$$(\star) \qquad g(\hat{1}) = \hat{3}, \;\; g(\hat{2}) = \hat{2} \;\text{ and }\; g(\hat{4}) = \hat{0}.$$

How many distinct polynomials of degrees ≤ 4 *satisfying the conditions* $(\star)$ *exist in* $\mathbf{Z_5}[x]$ *? How many of these polynomials satisfy an additional condition that* $g(\hat{0}) = \hat{0}$ *? List two different polynomials of degree* ≤ 4 *that satisfy the conditions* $(\star)$ *and* $g(\hat{0}) = \hat{0}$.

Exercise 2. *Show that* $x^3 - 2$ *is an irreducible polynomial in* $\mathbf{Q}[x]$. *Describe the elements of the field* $K = \mathbf{Q}[x]/\langle x^3 - 2\rangle$. *Find the (multiplicative) inverse of the element* $\overline{x}$ *and the (multiplicative) inverse of the element* $\overline{2x^2 - 1}$ *in* K.

Exercise 3. *How many (distinct) equivalences there exist on a set of three elements ?*

Exercise 4. *Given finite sets* A, B *and* C, *derive a formula for the number of the elements that belong to one, but not more than one, of the sets* A, B, C *in terms of numbers of elements in the sets* A, B, C *and in their intersections.*

Exercise 5. *Show that the relation* ∇ *on the set* $\mathbf{N}$ *of natural numbers defined by*

$$a \nabla b \;\; \Leftrightarrow \;\; a = 3^k b \;\text{ for }\; k \in \mathbf{Z}$$

is an equivalence and describe the equivalence class containing the number 135.

Exercise 6. *Denote by* $\mathbf{R}^*$ *the set of all positive real numbers. Consider the following relations* $\bowtie$ *on* $A = \mathbf{R}^* \times \mathbf{R}^*$, *show that they are equivalences and, in each case, describe the equivalence classes:*

(i) $(a, b) \bowtie (c, d)$ *if and only if* $ad = bc$;
(ii) $(a, b) \bowtie (c, d)$ *if and only if* $ab = cd$;
(iii)$(a, b) \bowtie (c, d)$ *if and only if* $a^2 + b^2 = c^2 + d^2$;
(iv) $(a, b) \bowtie (c, d)$ *if and only if* $a - b = c - d$ *and*
(v) $(a, b) \bowtie (c, d)$ *if and only if* $a + b = c + d$.

Solutions will be sent to all students by e-mail.
They will be also available in the display case opposite of my office 4205HP
on Monday, March 14, 2005.