

HOMEWORK No 6 (March 2, 2005)

Exercise 1. In $\mathbf{Z}_7[x]$, find the greatest common divisor $D(x)$ of the polynomials $f(x) = \hat{6}x^5 + \hat{4}x^3 + x^2 + \hat{3}$ and $g(x) = x^7 + \hat{3}x^5 + \hat{5}x^3 + x + \hat{2}$.

Find $u(x)$ and $v(x)$ in $\mathbf{Z}_7[x]$ such that $f(x)u(x) + g(x)v(x) = D(x)$.

Exercise 2. Determine the polynomial $f(x) \in \mathbf{Z}_5[x]$, $\deg f \leq 4$, such that

$$f(\hat{0}) = \hat{4}, f(\hat{1}) = \hat{4}, f(\hat{2}) = \hat{4}, f(\hat{3}) = \hat{0}, \text{ and } f(\hat{4}) = \hat{2}.$$

Exercise 3. (a) Find all $n \in \mathbf{N}$ such that $n^2 + n + 1$ is a multiple of 13.

(b) Using the fact that in $\mathbf{Z}_4[x]$, $2(x + \hat{1}) = 2(x + \hat{3})$, describe the set of all polynomials $f(x) \in \mathbf{Z}_4[x]$ of degree ≤ 3 such that $f(x) = \hat{0}$ for all $x \in \mathbf{Z}_4$.

(c) Show that there is an infinite number of polynomials $f \in \mathbf{Z}_4[x]$ such that $f^2 = 1$. [Hint: $f(x) = \hat{2}x + \hat{3}$ is such a polynomial.]

Exercise 4. Consider the subring $\mathbf{Z}[x] \subset \mathbf{Q}[x]$ of all polynomials of $\mathbf{Q}[x]$ with integral coefficients. Define the mapping $\phi : \mathbf{Z}[x] \rightarrow \mathbf{Z}_n[x]$ by

$$\phi\left(\sum_t a_t x^t\right) = \sum_t \hat{a}_t x^t,$$

where \hat{a}_t is the congruence class containing $a_t \in \mathbf{Z}$.

(a) Show that ϕ is a homomorphism, and describe its kernel.

(b) Prove that if $f \mid g$ in $\mathbf{Z}[x]$, then $\phi(f) \mid \phi(g)$ in $\mathbf{Z}_n[x]$.

(c) Give an example when $\phi(f) \mid \phi(g)$ in $\mathbf{Z}_n[x]$, but f is not a divisor of g in $\mathbf{Z}[x]$.

Exercise 5. (a) Show that $f(x) = x^2 + x + \hat{1}$ is an irreducible polynomial of $\mathbf{Z}_5[x]$.

(b) Construct the field

$$\frac{\mathbf{Z}_5[x]}{\langle x^2 + x + \hat{1} \rangle}.$$

(c) Show that this field has $5^2 = 25$ elements and find the (multiplicative) inverse of the element $x + \hat{3}$.

Solutions will be sent to all students by e-mail.

They will be also available in the display case opposite of my office 4205HP
on Monday, March 7, 2005.