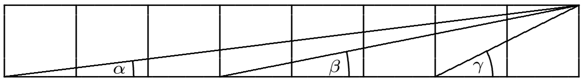
HOMEWORK No 5 (February 9, 2005)

Exercise 1. *Denote by r and α the modulus and the argument of a non-zero complex number $z = x+iy$, respectively. Show that all the following complex numbers have modules r and express their arguments in terms of α:*
$x-iy$; $-x+iy$; $-x-iy$; $y+ix$; $-y+ix$; $y-ix$; $-y-ix$.

Exercise 2. *Calculate the algebraic form of the number $(1+i\sqrt{3})^7 \times (1-i\sqrt{3})^{-12}$ using its trigonometric form.*

Exercise 3. *In the following "8 squares display" prove that $\alpha+\beta+\gamma = \frac{\pi}{4}$, and calculate an approximate value of π.*



Exercise 4. (a) *Let $z = x+iy$ and $z^2 = a+ib$ be algebraic forms of z and z^2. Show that if $|z| = 1$, then $x^2 = \frac{1}{2}(1+a)$ and $y^2 = \frac{1}{2}(1-a)$.*
(b) *Put $x = \cos\frac{\pi}{8}$ and $y = \sin\frac{\pi}{8}$. Find the trigonometric form of z^2 and thus calculate $\cos\frac{\pi}{8}$ and $\sin\frac{\pi}{8}$.*

Exercise 5. *Let α be a real number $0 \le \alpha \le \pi$.* (a) *Calculate $e^{i\alpha} + e^{-i\alpha}$ and $e^{i\alpha}\, e^{-i\alpha}$.*
(b) *Solve the equation $(z^2 - 2(\cos\alpha)z + 1)(z^2 + 2(\cos\alpha)z + 1) = 0$, and show that the images of the solutions are the vertices of a rectangle.*

Exercise 6. *Put $\omega = e^{i\frac{2\pi}{5}}$.* (a) *Calculate ω^5 and prove that $1+\omega+\omega^2+\omega^3+\omega^4 = 0$.*
(b) *Let $u = \omega+\omega^4$ and $v = \omega^2+\omega^3$. Calculate $u+v$ and uv and deduce that u and v are solutions of a quadratic equation. Solve this equation.*
(c) *Justify the equality $u = \omega+\overline{\omega}$ and calculate $\cos\frac{2\pi}{5}$.*

Exercise 7. *Let z_1, z_2, z_3 be three distinct complex numbers such that $|z_1| = |z_2| = |z_3|$. Prove that the following three statements are equivalent:*
(a) *z_1, z_2, z_3 are the vertices of an equilateral triangle*;
(b) *$z_1+z_2+z_3 = 0$;*
(c) *z_1, z_2, z_3 are the roots of an equation $z^3 = w$, where $w \in \mathbf{C}$.*

BONUS QUESTION. *In analogy to Exercise 7., prove, for $z_1, z_2, z_3, z_4 \in \mathbf{C}$ such that $|z_1| = |z_2| = |z_3| = |z_4|$,*

$$z_1+z_2+z_3+z_4 = 0 \textit{ if and only if } z_1, z_2, z_3, z_4 \textit{ are the vertices of a rectangle.}$$

Solutions will be sent to all students by e-mail.

They will be also available in the display case opposite of my office 4205HP

on Monday, February 14, 2005.