HOMEWORK No 3 (January 26, 2005)

Exercise 1. *Prove that, for any natural* n*,*

(i) $5^n - 1$ *and* $13^n - 1$ *are divisible by* 4*;*

(ii) *in general, for all natural* k*,* $(4k+1)^n - 1$ *is divisible by* 4*.*

[Prove by (a) using summation of a geometric series; (b) applying induction; (c) using congruence.]

Exercise 2. (a) *Show that* $2^n - 1$ *is a multiple of* 17 *if* $n = 8k, k \in \mathbf{N}$*.*

(b) *Using Euclidean division* $n = 8k + r$*,* $0 \leq r < n$*, prove that* $2^n \equiv 2^r \pmod{17}$*.*

(c) *Determine all values of the powers of* $\alpha_2 = [2]$ *in* $\mathbf{Z_{17}}$*.*

Exercise 3. (a) *Express all numbers between one and twenty in the binary and tertiary systems.*

(b) *Write down the number "two thousand and four" in the* (i) *decimal ;* (ii) *binary ;* (iii) *5-ary and (iv) 12-ary systems.*

(c) *Express the following numbers in the decimal system:* (i) $(111\ 111\ 111)_2$ *(binary system) ;* (ii) $(10642)_7$ *and* (iii) $(10\alpha 0\beta)_{12}$*.*

Exercise 4. (a) *Solve the diophantine equation*

$$(\star) \qquad 461267x + 1612151y = 45670.$$

(b) *Show that there is no solution of* $(\star)$ *such that both* x *and* y *are positive.*

(c) *Find the (multiplicative) inverse of* $\alpha_{101} = [101]$ *in* $\mathbf{Z_{353}}$*.*

(d) *Solve the linear equation* $\alpha_{101} \otimes x = \alpha_{100}$ *(i.e.* $[101] \otimes x = [100]$*) in* $\mathbf{Z_{353}}$*.*

(e) *Show that* α_{347} $(= [347])$ *is a solution of the quadratic equation*

$$(\star\star) \qquad x^2 \oplus \alpha_{101} \otimes x \oplus \alpha_{217} = \alpha_0 \ \text{ (i. e. } x^2 \oplus [101] \otimes x \oplus [217] = [0]) \ \text{ in } \mathbf{Z_{353}}.$$

(f) *Find the other solution of* $(\star\star)$ *in* $\mathbf{Z_{353}}$*.*

Exercise 5. *An astronomer observed on the day* J_0 *a celestial body* A *that has been appearing periodically every* 105 *days. After six days (on* $J_0 + 6$*), he observed another body* B *that has been appearing periodically every* 81 *days. Denote the day that both* A *and* B *appear simultaneously by* J_1*.*

(a) *Show that, in order to determine the date of the day* J_1*, one needs to solve the diophantine equation*

$$(\star\star\star) \qquad 35x - 27y = 2.$$

(b) *Write down a general solution of* $(\star\star\star)$*.*

(c) *How many days passed between* J_0 *and* J_1 *? If the date of* J_0 *was December 7, 1999, what was the date of* J_1 *?* [Note that 2000 is a leap year.] *How many days will have the astronomer to wait till the bodies* A *and* B *appear again the same day ?*

Solutions will be sent to all students by e-mail and will be available in the display case opposite of my office 4205HP

on Monday, January 31, 2005.