HOMEWORK No 10 (March 30, 2005)

Exercise 1. *Let R be the set of all real numbers with the following operations of "addition" $\oplus$ and "multiplication" $\odot$: $x \oplus y = x + y + 1$ and $x \odot y = xy + x + y$. Prove that $(R, \oplus, \odot)$ is a field with "zero" -1 and "identity" 0.*

Exercise 2. *Show that the Exercise 1. is a special case of the following theorem.*

Let F be a field and $a \in F, b \in F$ arbitrary elements such that $a \neq b$. Define the following two operations (of addition and multiplication) in the set $R = F$:

$$x \oplus y = x + y - a \quad \text{and} \quad x \odot y = \frac{1}{b-a}(xy - a(x+y) + ab).$$

Then, $(R, \oplus, \odot)$ is a field isomorphic to F (with "zero" a and "identity" b). Prove the theorem and formulate the special cases : (**i**) *zero $= 0$, identity $= b \neq 0$;* (**ii**) *zero $= a \neq 1$, identity $= 1$ and* (**iii**) *zero $= 1$, identity $= 0$.*

Exercise 3. *Choose an orthonormal bases (**1, i, j, k**) in the vector space $\mathbf{Q} = \mathbf{R}^4$, and define on $\mathbf{Q}$ multiplication by the following table*

1	**i**	**j**	**k**
i	**−1**	**k**	**−j**
j	**−k**	**−1**	**i**
k	**j**	**−i**	**−1**

Show that $\mathbf{Q}$ is a division ring (non-commutative field of **quaternions***).*

Exercise 4. (a) *Show that the set of all invertible elements in a ring forms a multiplicative group.*
(b) *If ab is a divisor of zero, then a or b is a divisor of zero.*
(c) *Find all invertible elements and all zero divisors in the ring $\mathbf{Z_{100}}$.*

Exercise 5. *Defining addition of two endomorphisms ϕ and ψ by $(\phi + \psi)(x) = \phi(x) + \psi(x)$ and multiplication by composition of mappings, prove that the set $E(G)$ of all endomorphisms of an abelian group G forms a ring. List the elements of $E(\mathbf{Z_4})$ and $E(\mathbf{Z_2} \times \mathbf{Z_2})$, and give the addition and multiplication tables for these rings.*

Solutions will be sent to all students by e-mail.
They will be also available in the display case opposite of my office 4205HP
on Monday, April 4, 2005.